Lesson - Quadratic equations and graphs

Year: Year 10

A quadratic expression has x^2 as its highest power, such as $x^2 - 5x + 6$. Quadratic equations and their parabola graphs are one of the most common GCSE topics. Foundation students solve quadratics by factorising and sketch parabolas. Higher students also factorise $ax^2 + bx + c$, complete the square, use the quadratic formula and find turning points.

Starter activity

Throw a soft ball upwards, or watch a basketball shot, and describe the path it takes. Is the path a straight line? Where is the highest point? Is the path symmetrical, and where is the line of symmetry? Now imagine a table of heights at one-second intervals: 0, 15, 20, 15, 0. Plot those points by hand on squared paper (time across, height up) and join them with a smooth curve. What is the name of the shape you have drawn?

Solving by factorising

Definition: quadratic equation

A **quadratic equation** can be written $ax^2 + bx + c = 0$ with $a \neq 0$. Its solutions are called **roots**.

Method: solving by factorising

1. Rearrange so that one side is 0: $ax^2 + bx + c = 0$.
2. Factorise.
3. Use the fact that if $A \times B = 0$ then $A = 0$ or $B = 0$.

Worked example 1

$x^2 - 5x + 6 = 0$.

Numbers with product 6 and sum -5 are -2 and -3. $(x - 2)(x - 3) = 0$, so $x = 2$ or $x = 3$.

Worked example 2

$x^2 = 5x - 4$.

Rearrange to $x^2 - 5x + 4 = 0$. $(x - 1)(x - 4) = 0$, so $x = 1$ or $x = 4$. Never divide both sides by x, because that loses a solution.

Method: solving by factorising

Special cases. $x^2 - 49 = 0$ gives $x = \pm 7$. $x^2 - 4x = 0$ gives $x(x - 4) = 0$, so $x = 0$ or $x = 4$.

Quadratic graphs

Definition: parabola

A **parabola** is the smooth, symmetrical curve that is the graph of a quadratic. It has a turning point, which is a minimum or a maximum.

Key fact: quadratic graphs

The graph of $y = ax^2 + bx + c$ is a smooth curve called a **parabola**. If $a > 0$ it is $\cup$-shaped with a minimum point; if a it is $\cap$-shaped with a maximum point. To draw one, complete a table of values and plot a smooth curve (not straight segments, no flat bottom).

For $y = x^2 - 4x$, the values for $x = 0, 1, 2, 3, 4$ are $0, -3, -4, -3, 0$.

Features of a parabola

Key fact: features of a parabola

- **Roots (x-intercepts):** where $y = 0$, found by solving the equation.
- **y-intercept:** put $x = 0$, giving c.
- **Line of symmetry:** halfway between the roots.
- **Turning point:** on the line of symmetry.

For $y = x^2 - 2x - 3$: roots from $(x - 3)(x + 1) = 0$ are $x = 3$ and $x = -1$; y-intercept -3; line of symmetry $x = \frac{3+(-1)}{2} = 1$; at $x = 1$, $y = 1 - 2 - 3 = -4$, so the minimum point is $(1, -4)$.

Quadratics in context

Key fact: quadratics in context

The area of a rectangle with sides x and $x + 3$ is 28. $x(x + 3) = 28$, $x^2 + 3x - 28 = 0$, $(x + 7)(x - 4) = 0$. Since a length cannot be negative, $x = 4$, so the rectangle is 4 by 7. A ball's height $h = 20t - 5t^2$ metres is 0 when $5t(4 - t) = 0$, so it lands at $t = 4$ s; its maximum is at $t = 2$: $h = 40 - 20 = 20$ m. Always reject solutions that do not make sense in the context.

Factorising when $a \neq 1$ (Higher tier)

Method: factorising when a is not 1 (Higher tier)

Solve $2x^2 + 5x - 3 = 0$. $ac = -6$; numbers 6 and -1.
$2x^2 + 6x - x - 3 = 2x(x+3) - 1(x+3) = (2x-1)(x+3)$. So $x = \frac{1}{2}$ or $x = -3$.

Completing the square (Higher tier)

Rule: completing the square (Higher tier)

Write $x^2 + bx + c$ as $(x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 + c$. For $x^2 + 6x + 2 = (x+3)^2 - 9 + 2 = (x+3)^2 - 7$. The turning point of $y = (x+p)^2 + q$ is $(-p, q)$, so here $(-3, -7)$. To solve $x^2 + 6x + 2 = 0$: $(x+3)^2 = 7$, $x = -3 \pm \sqrt{7}$. For $x^2 - 8x + 3 = (x-4)^2 - 13$, the minimum value is -13 at $x = 4$.

The quadratic formula (Higher tier)

Definition: discriminant (Higher tier)

The **discriminant** is $b^2 - 4ac$. If it is positive there are two real roots, if it is zero there is one, and if it is negative there are none.

Theorem: the quadratic formula (Higher tier)

For $ax^2 + bx + c = 0$: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Solve $2x^2 - 3x - 4 = 0$: $a = 2$, $b = -3$, $c = -4$. $x = \dfrac{3 \pm \sqrt{9 + 32}}{4} = \dfrac{3 \pm \sqrt{41}}{4}$, giving $x = 2.35$ or $x = -0.85$ to 2 d.p. Put the whole of $-b$ in brackets: $-(-3) = 3$. The expression $b^2 - 4ac$ is the **discriminant**: if it is negative there are no real solutions. For $x^2 + 2x + 5 = 0$: $4 - 20 = -16$, so no real solutions.

More worked examples in exam style

Worked example 3

Solve $x^2 + 3x = 10$ [3 marks].

Rearrange: $x^2 + 3x - 10 = 0$. Factorise $(x+5)(x-2) = 0$. So $x = -5$ or $x = 2$.

Worked example 4

The graph of $y = x^2 - 4x + 3$ crosses the x-axis at $x = 1$ and $x = 3$. Write down the equation of the line of symmetry and the coordinates of the minimum point [3 marks].

Line of symmetry $x = 2$ (midpoint of 1 and 3). At $x = 2$, $y = 4 - 8 + 3 = -1$. Minimum point $(2, -1)$.

Worked example 5 (Higher tier)

Solve $3x^2 - 5x - 2 = 0$ by factorising [3 marks].

$ac = -6$; numbers -6 and 1.
$3x^2 - 6x + x - 2 = 3x(x - 2) + 1(x - 2) = (3x + 1)(x - 2)$. $x = -\frac{1}{3}$ or $x = 2$.

Worked example 6 (Higher tier)

Use the quadratic formula to solve $x^2 - 4x - 3 = 0$ to 2 d.p. [3 marks].

$x = \dfrac{4 \pm \sqrt{16 + 12}}{2} = \dfrac{4 \pm \sqrt{28}}{2}$, so $x = 4.65$ or $x = -0.65$.

Common mistakes

Watch out

- Not rearranging to equal zero before factorising.
- Writing only one solution; there are usually two.
- Drawing a pointed or flat-bottomed parabola.
- Sign errors in the formula: $-b$ when b is negative, and $-4ac$ when c is negative.
- Giving a turning point as (p, q) instead of $(-p, q)$.

Exam technique

Exam technique

- Show the factorisation: a correct answer with no working may not earn all the marks.
- Check solutions in the original equation.
- When sketching, label the intercepts and turning point, and draw a smooth curve.
- If the question gives a context, say which solution you reject and why.

Quadratic equations and their pictures

A quadratic expression has an x^2 term as its highest power, and its graph is a smooth symmetrical curve called a parabola. When the coefficient of x^2 is positive the curve is U-shaped with a minimum, and when it is negative the curve is upside down with a maximum. The solutions of a quadratic equation are the x-values where the graph crosses the x-axis, called the roots or zeros. This links algebra and graphs: factorising to solve $x^2 - 2x - 3 = 0$ gives $(x - 3)(x + 1) = 0$, so the curve crosses the axis at $x = 3$ and $x = -1$. The line of symmetry lies halfway between the two roots, and the turning point lies on that line, so you can find it by substituting the x value of the line of symmetry into the equation of the curve.

Worked example 7: finding the turning point

Find the turning point of $y = x^2 - 8x + 7$ using the roots [3 marks].

Factorise: $y = (x - 1)(x - 7)$, so the roots are $x = 1$ and $x = 7$. The line of symmetry is halfway between them: $x = \dfrac{1 + 7}{2} = 4$. Substitute: $y = 16 - 32 + 7 = -9$. The turning point is $(4, -9)$ and it is a minimum because the x^2 coefficient is positive. You can also use completing the square: $y = (x - 4)^2 - 9$ gives the same point.

Worked example 8: reading a graph for an equation

The graph of $y = x^2 - 4x - 1$ is drawn. Explain how to use it to solve $x^2 - 4x - 1 = 3$ [2 marks].

Draw the horizontal line $y = 3$ on the same axes. The x-values of the points where the line crosses the curve are the solutions of $x^2 - 4x - 1 = 3$. This works because on the curve $y = x^2 - 4x - 1$ and on the line $y = 3$, so where they meet $x^2 - 4x - 1 = 3$. The solutions are $x = 2 \pm \sqrt{8}$, about -0.8 and 4.8.

Worked example 9: a quadratic with no real solutions

Show that $x^2 + 2x + 5 = 0$ has no real solutions [3 marks].

The discriminant is $b^2 - 4ac = 4 - 20 = -16$, which is negative, so there are no real roots. By completing the square, $x^2 + 2x + 5 = (x + 1)^2 + 4$. A squared bracket is never negative, so the smallest value of the expression is 4, which means the graph stays above the x-axis and never crosses it.

Worked example 10: a problem in context

A rectangle is 3 cm longer than it is wide. A square of side 2 cm is removed from one corner, leaving an area of 36 cm². Find the width of the rectangle [4 marks].

Let the width be x. The rectangle has area $x(x + 3)$. Remove 4 cm²: $x(x + 3) - 4 = 36$,

so $x^2 + 3x - 40 = 0$. Factorise: $(x + 8)(x - 5) = 0$. The width is positive, so $x = 5$ cm. Check: the rectangle is $5 \times 8 = 40$, less 4 gives 36. ✓

Note: choosing a method

If the question says “by factorising”, you must factorise. If it asks for answers “to 2 decimal places” or “in surd form”, it expects the quadratic formula or completing the square. If the quadratic does not seem to factorise, check the discriminant $b^2 - 4ac$: it is a perfect square only if the equation factorises with whole numbers. Always rearrange so that one side is zero before using any method.

Watch out: lost solutions

When solving $x^2 = 9x$ never divide both sides by x, because you lose the solution $x = 0$. Instead write $x^2 - 9x = 0$ and factorise to $x(x - 9) = 0$. Also remember to reject a negative answer when x is a length.

Did you know?

The path of a thrown ball is a parabola, and Galileo Galilei was the first to prove this, around 1608, by rolling balls down ramps and measuring where they landed. The word *parabola* comes from the Greek for “thrown alongside”. Parabolas also appear in car headlights and satellite dishes, because a parabolic mirror reflects all parallel rays to a single point, called the focus. The quadratic formula goes back almost 4000 years: Babylonian scribes solved problems like “find the sides of a rectangle with a given area and perimeter” with a method that is equivalent to our formula, although they only used positive numbers.

Otto's tip

Check both solutions by substituting them back. If you are using a calculator on the formula, store $b^2 - 4ac$ first so that you do not mistype the square root.