# Lesson - Linear equations and inequalities

Year: Year 10

Solving equations is one of the most useful GCSE skills. You need to solve linear equations with the unknown on one or both sides, with brackets and fractions, form equations from words or diagrams, and solve and represent inequalities. Inequalities in two variables, quadratic inequalities and set notation are Higher tier only.

**Starter activity**

Mia and Joe each think of a number. Mia doubles hers and adds 5. Joe adds 8 to his. They end up with the same number. Joe's number is 3 more than Mia's. Can you use this information to find both numbers by trial and improvement? Next, call Mia's number $m$ and write what you know as an equation. Does solving it give the same answer as your trials? Why is the algebra method more reliable if the numbers are awkward?

## The balance method

**Definition: equation**

An **equation** says that two expressions are equal. To solve it, find the value of the unknown that makes it true.

**Method: the balance method**

An equation is like a balance: whatever you do to one side, do to the other. Undo operations in the reverse order. For $3x + 5 = 20$: subtract 5 to get $3x = 15$, then divide by 3 to get $x = 5$. Check by substituting back: $3 \times 5 + 5 = 20$.

## Unknown on both sides

**Method: unknown on both sides**

Collect the $x$ terms on one side (choose the side that keeps the coefficient positive) and the numbers on the other. $5x - 7 = 3x + 9$: subtract $3x$ to get $2x - 7 = 9$; add 7 to get $2x = 16$; so $x = 8$.

## Brackets and fractions

**Method: brackets and fractions**

Expand brackets first. $4(x-3)=2(x+5)$ gives $4x-12=2x+10$, so $2x=22$ and $x=11$.

For fractions, multiply both sides by the denominator. $\frac{x+3}{2}=7$ gives $x+3=14$, so $x=11$. With fractions on both sides, cross-multiply: $\frac{2x-1}{3}=\frac{x+4}{2}$ gives $2(2x-1)=3(x+4)$, so $4x-2=3x+12$ and $x=14$. When there are several fractions in a sum, multiply every term by the lowest common denominator: $\frac{x+1}{3}-\frac{x-2}{4}=1$ becomes $4(x+1)-3(x-2)=12$, so $x+10=12$ and $x=2$.

## Forming and solving equations

**Method: forming and solving equations**

Read the problem, define the unknown ("let $x$ be ..."), write an equation, solve it and answer in context.

- Angles in a triangle $x$, $2x+10$ and $3x-10$ add up to 180: $6x=180$, $x=30$. The angles are 30, 70 and 80 degrees.
- A rectangle has sides $(x+3)$ cm and $(2x-1)$ cm and perimeter 40 cm. $2(3x+2)=40$, so $3x+2=20$, $x=6$. The sides are 9 cm and 11 cm.

## Inequalities

**Definition: inequality**

An **inequality** compares two quantities using $<$, $>$, $\leq$ or $\geq$. Its solution is usually a range of values.

**Rule: solving inequalities**

The signs are  (less than), $>$ (greater than), $\leq$ (less than or equal to) and $\geq$ (greater than or equal to). Solve them like equations with one rule: **if you multiply or divide by a negative number, reverse the inequality sign**. For $7-3x>1$: $-3x>-6$; divide by $-3$ and reverse: $x$. An easier method that avoids the reversal is to add $3x$ to both sides: $7>1+3x$, $6>3x$, $2>x$.

## Number lines and integer solutions

**Rule: number lines and integer solutions**

On a number line use an **open circle** for  or $>$ and a **solid (filled) circle** for $\leq$ or $\geq$. The inequality $-1\leq x$ has a solid circle at $-1$ and an open circle at 4 with the line between them. To list integer solutions of $-3$: divide by 2 to get $-1.5$, so the integers

are $-1, 0, 1, 2, 3$.

## Quadratic inequalities (Higher tier)

**Method: quadratic inequalities (Higher tier)**

Solve the related equation, sketch the parabola and read off where it is above or below the $x$-axis. $x^2$: roots $\pm 4$, so $-4$. $x^2 - x - 6 > 0$: factorise $(x - 3)(x + 2) > 0$, roots 3 and $-2$. The graph is above the axis outside the roots: $x$ or $x > 3$.

## Inequalities in two variables (Higher tier)

**Definition: inequality**

An **inequality** compares two quantities using $<$, $>$, $\leq$ or $\geq$. Its solution is usually a range of values.

**Method: inequalities in two variables (Higher tier)**

Draw each boundary line (solid for $\leq$ or $\geq$, dashed for  or $>$), test a point to decide which side to shade, and describe the region that satisfies all the inequalities. The triangle with vertices $(0, 0)$, $(0, 4)$ and $(4, 4)$ is defined by $x \geq 0$, $y \leq 4$ and $y \geq x$.

## Worked example with unknown on both sides and brackets

**Worked example 1**

**Solve $3(2x - 1) = 2(x + 5)$ [3 marks].**

Expand: $6x - 3 = 2x + 10$. Subtract $2x$: $4x - 3 = 10$. Add 3: $4x = 13$. Divide: $x = \frac{13}{4} = 3.25$. Check: left $= 3(6.5 - 1) = 16.5$, right $= 2(8.25) = 16.5$. Equal.

## More worked examples in exam style

**Worked example 2**

**Solve $\frac{5x - 2}{4} = 3x - 5$ [3 marks].**

Multiply both sides by 4: $5x - 2 = 12x - 20$. Collect: $18 = 7x$, so $x = \frac{18}{7}$.

**Worked example 3**

**Ella is $x$ years old. Her brother is 4 years older and her mother is three times as old as Ella. The sum of the three ages is 79. Find Ella's age [4 marks].**

$x + (x + 4) + 3x = 79$, $5x + 4 = 79$, $5x = 75$, $x = 15$.

### Worked example 4

**Solve $-4$ and list the integer solutions [3 marks].**

Subtract 2: $-6$. Divide by 3: $-2$. Integers: $-1, 0, 1, 2, 3$.

## Common mistakes

### Watch out

- Forgetting to apply the operation to every term on both sides.
- Not reversing the sign when dividing an inequality by a negative number.
- Using a solid circle when the inequality is strict.
- Giving a decimal as an integer solution.
- Not multiplying *every* term by the denominator in equations with fractions.

## Exam technique

### Exam technique

- Keep the equals signs lined up and do one step per line.
- Use a number line to check inequality answers: pick a number from the shaded part.
- In word problems, define the unknown clearly and answer the original question.

## Equations and the balance idea

An equation is a statement that two expressions are equal, and it is like a pair of scales that balance. Whatever you do to one side you must do to the other, so the scales stay balanced. To solve an equation you build up a sequence of operations that undo what has been done to the unknown, and you work from the outside in. The most reliable habit is to keep each step on a new line with the equals signs lined up, and to substitute your answer back into the original equation. Inequalities behave in almost exactly the same way, except that the equals sign is replaced by $<$, $>$, $\leq$ or $\geq$, and there is one rule to remember: when you multiply or divide both sides by a negative number the inequality sign must be reversed.

### Worked example 6: brackets on both sides

**Solve $3(2x - 1) - 2(x - 4) = 5x + 9$ [4 marks].**

Expand the brackets carefully: $6x - 3 - 2x + 8 = 5x + 9$. Collect like terms on the left: $4x + 5 = 5x + 9$. Subtract $4x$ from both sides: $5 = x + 9$. Subtract 9: $x = -4$. Check: left side $3(-8 - 1) - 2(-4 - 4) = 3(-9) - 2(-8) = -27 + 16 = -11$ and right side $5(-4) + 9 = -11$. They match, so $x = -4$ is correct.

### Worked example 7: an inequality with a negative coefficient

**Solve $7 - 3x > 22$ and show the solution on a number line [3 marks].**

Subtract 7 from both sides: $-3x > 15$. Divide both sides by $-3$ and reverse the sign: $x < -5$. On a number line draw an *open* circle at $-5$ (because $-5$ is not included) and an arrow pointing to the left. Check by testing $x = -6$: $7 - 3(-6) = 25$, which is greater than 22. ✓

### Worked example 8: forming an equation from a diagram

**The sides of a triangle are $x$ cm, $(x + 4)$ cm and $(2x - 3)$ cm. The perimeter is 33 cm. Find the longest side [3 marks].**

Perimeter: $x + (x + 4) + (2x - 3) = 33$, so $4x + 1 = 33$ and $x = 8$. The sides are 8 cm, 12 cm and 13 cm. The longest side is 13 cm. Check: $8 + 12 + 13 = 33$ ✓. Always find all the quantities and check them against the question, because the question asks for the longest side and not for $x$.

### Worked example 9: equations with the unknown in the denominator of a fraction

**Solve $\frac{5}{x} = \frac{2}{x - 3}$ [3 marks].**

Cross-multiply, which means multiply both sides by $x(x - 3)$: $5(x - 3) = 2x$. Expand: $5x - 15 = 2x$. Then $3x = 15$ and $x = 5$. Check: $\frac{5}{5} = 1$ and $\frac{2}{5 - 3} = 1$. ✓ Note that the solution cannot be $0$ or $3$ because the fractions would not be defined.

### Note: the number line

On a number line an open circle means the number is not included, which is used for $<$ and $>$. A closed (filled) circle means the number is included, which is used for $\leq$ and $\geq$. A double inequality such as $-2 < x \leq 4$ is shown by a line between the two circles. Integer solutions are counted from this line, so always check whether the end values are included.

### Watch out: common errors

The most common errors are adding instead of subtracting when moving a term, forgetting to multiply *every* term when clearing fractions, and failing to reverse the sign when dividing an inequality by a negative number. Test a value from your

solution set to check.

**Did you know?**

The idea of an equation is very old. The Rhind papyrus, written in Egypt about 1650 BC, contains problems such as “a quantity and a seventh of it make 19”, which are linear equations, solved by a method called false position. The word “equation” itself comes from the Latin *aequare*, “to make equal”. The symbols for inequalities, $<$ and $>$, were first used by the English mathematician Thomas Harriot in a book published in 1631 after his death. Today equations are used in every area of life: to plan a school trip budget, to calculate medicine doses, to design a bridge and to program the satnav that finds the fastest route.

**Otto's tip**

After solving, always substitute your answer back into the original equation. For an inequality, test one value inside your solution and one outside.