Lesson - Fractions, decimals and percentages

Year: Year 10

This topic is worth a large share of the marks on both GCSE tiers. You must be able to move easily between fractions, decimals and percentages, and to use each form as an operator (a "fraction of", a "percentage of", a percentage change). Questions come up on Paper 1 (non-calculator) and Papers 2 and 3.

Starter activity

A trainer costs £80 in two shops. Shop A says “30% off”. Shop B says “10% off, then a further 20% off”. A friend says, “Both come to 30% off, so the price is the same.” Work out each final price and find out who is right. Then change the order in shop B (20% off first, then 10%) and see what happens. What does this tell you about percentages that follow one another?

Ordering and equivalent forms

Definition: equivalent forms

A fraction, a decimal and a percentage are three ways of writing the same number. $\frac{3}{4} = 0.75 = 75\%$. A percentage is a fraction out of 100.

Key fact: equivalent fractions, decimals and percentages

To compare numbers in different forms, convert them all to decimals (or all to percentages). Use place value to line up decimals: write 0.4 as 0.40 so it can be compared with 0.41.

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{1}{5}$	0.2	20%
$\frac{1}{8}$	0.125	12.5%
$\frac{1}{3}$	0.333...	33.3...%

Learn these by heart: they save time on the non-calculator paper.

The four operations with fractions

Rule: the four operations with fractions

- **Add or subtract:** use a common denominator. $\frac{3}{4} + \frac{2}{5} = \frac{15}{20} + \frac{8}{20} = \frac{23}{20} = 1\frac{3}{20}$.
- **Multiply:** multiply the numerators and the denominators (cancel first if you can).
- **Divide:** keep the first fraction, flip the second, then multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.
- **Mixed numbers:** always change to improper fractions before multiplying or dividing. $2\frac{1}{3} = \frac{7}{3}$.

Converting between forms

Rule: converting between forms

Fraction to decimal: divide the numerator by the denominator. Decimal to percentage: multiply by 100. Decimal to fraction: use place value, then simplify, for example $0.625 = \frac{625}{1000} = \frac{5}{8}$.

Percentages as operators

Rule: percentages as operators

To find a percentage of an amount, convert to a decimal multiplier. 35% of £240 is $0.35 \times 240 = £84$. To write one number as a percentage of another, divide and multiply by 100: 18 out of 45 is $\frac{18}{45} \times 100 = 40\%$.

Percentage change and multipliers

Definition: multiplier

A **multiplier** is the number you multiply by to apply a change in one step. A 15% increase has multiplier 1.15 and a 12.5% decrease has multiplier 0.875.

Rule: multipliers and percentage change

A percentage increase of $p\%$ has multiplier $1 + \frac{p}{100}$; a decrease has multiplier $1 - \frac{p}{100}$. So +15% is $\times 1.15$ and -12.5% is $\times 0.875$.

Percentage change $= \dfrac{\text{change}}{\text{original}} \times 100$.

Reverse percentages: if the amount after a change is known, divide by the multiplier. A price after a 20% increase is 120% of the original, so original $= \text{new} \div 1.2$.

Method: percentage change and reverse percentages

1. Write the multiplier for the change ($1 + \frac{p}{100}$ or $1 - \frac{p}{100}$).
2. Going from the original to the new amount: multiply.
3. Going back to the original: divide by the multiplier.
4. For a percentage change, use $\frac{\text{change}}{\text{original}} \times 100$.

Recurring decimals (Higher tier)

Definition: recurring decimal

A **recurring decimal** has a digit or block of digits that repeats for ever. The dot notation marks the repeat: $0.\dot{3}\dot{6} = 0.363636\ldots$

Key fact: recurring decimals (Higher tier)

A recurring decimal can always be written as a fraction. Multiply by 10 if one digit repeats, by 100 if two repeat, and subtract to remove the repeating part. Examples: $0.\dot{7} = \frac{7}{9}$ and $0.1\dot{6} = \frac{1}{6}$. The Higher tier asks you to show this process clearly: write "let $x =$", multiply, subtract, then simplify.

Method: writing a recurring decimal as a fraction (Higher tier)

1. Write "let $x =$" the recurring decimal.
2. Multiply by 10 if one digit repeats, by 100 if two digits repeat.
3. Subtract the two equations to remove the repeating part.
4. Solve for x and simplify the fraction.

Worked examples

Worked example 1

Work out $3\frac{1}{2} \div 1\frac{3}{4}$.

$\frac{7}{2} \div \frac{7}{4} = \frac{7}{2} \times \frac{4}{7} = \frac{28}{14} = 2$.

Worked example 2

A coat costs £84 after a 20% increase. Find the original price.

The new price is 120% of the original. $84 \div 1.2 = £70$.

Worked example 3

Increase 40 to 52. Find the percentage change.

Change $= 12$. $\frac{12}{40} \times 100 = 30\%$ increase.

Worked example 4 (Higher tier)

Write $0.\dot{3}\dot{6}$ as a fraction.

Let $x = 0.363636\ldots$. Then $100x = 36.3636\ldots$. Subtract: $99x = 36$, so $x = \frac{36}{99} = \frac{4}{11}$.

More worked examples in exam style

Worked example 5 (adding mixed numbers)

Work out $2\frac{3}{5} + 1\frac{7}{10}$ [3 marks].

Add the whole numbers: $2 + 1 = 3$. Add the fractions with a common denominator: $\frac{6}{10} + \frac{7}{10} = \frac{13}{10} = 1\frac{3}{10}$. Total $= 4\frac{3}{10}$. Show each step: the method marks are for the common denominator and for combining the whole numbers.

Worked example 6 (fraction of an amount, then the rest)

A school has 640 pupils. $\frac{3}{8}$ travel by bus and 25% walk. The rest cycle. How many cycle? [4 marks].

Bus: $640 \div 8 \times 3 = 240$. Walk: $0.25 \times 640 = 160$. Cycle: $640 - 240 - 160 = 240$.

Worked example 7 (percentage of a percentage change chain)

A house price rises by 10% in year 1 and by 5% in year 2. What is the overall percentage increase? [3 marks].

Multiplier $1.1 \times 1.05 = 1.155$. The overall increase is 15.5%, not 15%.

Worked example 8 (comparing offers)

Shop A: $\frac{1}{3}$ off £90.

Shop B: 30% off £88. Which is cheaper? A: $90 \times \frac{2}{3} = £60$. B: $88 \times 0.7 = £61.60$. Shop A is cheaper by £1.60. Always finish with a statement answering the question.

Common mistakes

Watch out

- Adding denominators: $\frac{1}{2} + \frac{1}{3}$ is not $\frac{2}{5}$.
- Flipping the wrong fraction when dividing, or flipping a mixed number without

converting it first.
- Using the new value instead of the original in a percentage change.
- Treating a reverse percentage as "take off 20%" instead of dividing by 1.2.
- Two successive percentage changes cannot be added: +10% then -10% is $\times 1.1 \times 0.9 = 0.99$, a 1% decrease.

Exam technique

Exam technique

- On the non-calculator paper, convert awkward fractions to the nearest friendly ones ($\frac{1}{8} = 12.5\%$) before you start.
- In "show that" questions, write the full calculation and end with the number you were asked to show.
- Check that a percentage increase gives a bigger answer and a decrease a smaller one; this catches most multiplier slips.
- Give money answers to 2 decimal places, for example £61.60 not £61.6.

Choosing a method

Fractions, decimals and percentages are three ways of writing the same number, and the best exam technique is to pick the form that makes the calculation easiest. For simple fractions of an amount, divide by the denominator and multiply by the numerator. For percentages use the building blocks 10%, 5% and 1% in a non-calculator paper, and a single multiplier in a calculator paper. For percentage change, always divide the change by the *original* value. For repeated change, such as interest or depreciation, raise the multiplier to a power. Finally, whenever a question gives the value *after* a change, you are doing a reverse percentage: divide by the multiplier, never subtract the percentage.

Worked example 9: mixed numbers

Work out $2\frac{3}{5} + 1\frac{3}{4}$ without a calculator [3 marks].

Add the whole numbers: $2 + 1 = 3$. Add the fractions: $\frac{3}{5} + \frac{3}{4} = \frac{12}{20} + \frac{15}{20} = \frac{27}{20} = 1\frac{7}{20}$. Combine: $3 + 1\frac{7}{20} = 4\frac{7}{20}$. Check with decimals: $2.6 + 1.75 = 4.35$ and $\frac{7}{20} = 0.35$. The check confirms the answer.

Worked example 10: compound percentage with a table

£1500 is invested at 3.2% compound interest per year. Find the value after 3 years to the nearest penny [3 marks].

The multiplier is 1.032. Year 1: $1500 \times 1.032 = 1548$. Year 2: $1548 \times 1.032 = 1597.536$. Year 3: $1597.536 \times 1.032 = 1648.657$. The value is £1648.66. Using a power gives the

same: $1500 \times 1.032^3 = 1648.66$. Round only at the very end, or the rounding errors build up.

Worked example 11: percentage increase and decrease together

A house price rose by 12% and then fell by 5%. The final price is £213 750. Find the original price [3 marks].

The combined multiplier is $1.12 \times 0.95 = 1.064$. Original $= 213\,750 \div 1.064 = 200\,893$ to the nearest pound. The overall change is an increase of 6.4%, not 7%. Never combine the two percentages by simply adding or subtracting them.

Worked example 12: finding a percentage of a percentage

In a school 45% of the pupils are boys. Of the boys, 60% play a sport. What percentage of the whole school are boys who play a sport? [2 marks]

Multiply the two decimals: $0.45 \times 0.6 = 0.27$. So 27% of the whole school are boys who play a sport. If the school has 1200 pupils then $1200 \times 0.27 = 324$ pupils.

Note: percentage change formula

Percentage change $= \dfrac{\text{new value} - \text{original value}}{\text{original value}} \times 100$. A positive answer is an increase and a negative answer is a decrease. If a question asks for “percentage profit” or “percentage loss”, the original value is the cost price. If it asks for an “error” as a percentage, the original is the true value.

Watch out: the same percentage up and down

A 10% rise followed by a 10% fall does *not* return to the start, because the 10% fall is taken from the larger value. The overall multiplier is $1.1 \times 0.9 = 0.99$, a loss of 1%.

Did you know?

The word percent comes from the Latin *per centum*, meaning “by the hundred”. The percentage sign developed from a scribe’s shorthand in Italian manuscripts in the fifteenth century. Money is where percentages matter most: the Bank of England’s base rate, inflation, VAT, tax and mortgage rates are all stated as percentages. Compound interest was described by the mathematician Jacob Bernoulli in 1683. He asked what would happen if interest was paid every instant, and discovered the number $e \approx 2.718$, one of the most important numbers in all of mathematics. Albert Einstein is said to have called compound interest the most powerful force in the universe, although there is no evidence that he really said it.

Otto's tip

Write the multiplier first, then decide whether to multiply or divide. If you are going from the original to the new amount, multiply. If you are going back to the original, divide. Show the multiplier in your working to earn the method mark.

In the exam, always give fractions in their simplest form unless asked otherwise, and write the units (£, %) in your final answer.