Lesson - Formulae and identities

Year: Year 10

A formula is a rule that links quantities, such as $v = u + at$ or $A = \pi r^2$. At GCSE you must be able to substitute into formulae, rearrange them to change the subject, tell an identity from an equation, and use algebra to show that two expressions are the same. Algebraic proof is Higher tier only; the rest is on both papers.

Starter activity

The formula $C = 2\pi r$ gives the circumference of a circle. Measure the circumference and diameter of three round objects (a mug, a coin, a bin lid) and divide circumference by diameter each time. What do you notice? Now use the formula to predict the circumference of a circle of radius 20 cm and then work out what radius is needed for a circumference of 100 cm. Which of the two steps needed you to rearrange the formula?

Substituting into a formula

Definition: formula

A **formula** is a rule that links quantities using letters, such as $v = u + at$ or $A = \pi r^2$.

Method: substituting into a formula

Write the formula, replace each letter by its value (with units kept in mind), then calculate. For $v = u + at$ with $u = 5$, $a = 3$ and $t = 4$: $v = 5 + 3 \times 4 = 17$. For the trapezium $A = \frac{1}{2}(a + b)h$ with $a = 6$, $b = 10$, $h = 5$: $A = \frac{1}{2} \times 16 \times 5 = 40$. Do brackets first, then multiplication.

Changing the subject

Definition: subject of a formula

The **subject** of a formula is the letter on its own on one side. In $y = 3x + 5$ the subject is y.

Rule: changing the subject

Rearranging means getting a chosen letter on its own on one side. Treat it like solving an equation: whatever you do to one side you do to the other, using inverse operations in the reverse order.

- $y = 3x + 5$. Subtract 5: $y - 5 = 3x$. Divide by 3: $x = \dfrac{y - 5}{3}$.
- $C = 2\pi r$. Divide by 2π: $r = \dfrac{C}{2\pi}$.
- $P = 2(l + w)$. Divide by 2: $\frac{P}{2} = l + w$. Subtract w: $l = \frac{P}{2} - w$.
- $v^2 = u^2 + 2as$. Subtract u^2: $v^2 - u^2 = 2as$. Divide by $2s$: $a = \dfrac{v^2 - u^2}{2s}$.
- $y = 2x^2 + 1$. Subtract 1, divide by 2, take the square root: $x = \sqrt{\dfrac{y - 1}{2}}$ (positive root when $x > 0$).

Method: changing the subject

1. Decide which letter must end up on its own.
2. Undo the operations in the reverse order.
3. Do the same to both sides each time.
4. Check by substituting a simple pair of numbers.

When the subject appears twice (Higher tier)

Method: when the subject appears twice (Higher tier)

Collect all the terms containing the subject on one side, factorise, then divide. For $ax + b = cx + d$: $ax - cx = d - b$, so $x(a - c) = d - b$ and $x = \dfrac{d - b}{a - c}$. For $y = \dfrac{x + 1}{x - 2}$: multiply out, $y(x - 2) = x + 1$, expand, $xy - 2y = x + 1$, then $xy - x = 1 + 2y$, factorise $x(y - 1) = 1 + 2y$, and $x = \dfrac{2y + 1}{y - 1}$.

Expressions, equations, formulae and identities

Definition: expression, equation, formula and identity

An **equation** is true only for certain values: $2x + 1 = 7$ is true when $x = 3$. An **identity** is true for every value of the letter and uses the sign $\equiv$: $3(x + 2) \equiv 3x + 6$. To show an identity, expand or simplify one side until it matches the other. To check whether something is an equation or an identity, try a couple of different numbers.

Showing two expressions are equivalent

Key fact: showing two expressions are equivalent

Show that $(x + 2)^2 - x^2 \equiv 4(x + 1)$. Left side: $x^2 + 4x + 4 - x^2 = 4x + 4 = 4(x + 1)$.

Always start from one side and finish at the other, writing every line.

Algebraic proof (Higher tier)

Rule: algebraic proof (Higher tier)

Use letters to represent general numbers. An even number is $2n$, an odd number is $2n+1$, and consecutive integers are $n,\ n+1,\ n+2$. Then expand and simplify, and write a conclusion in words.

- **Sum of three consecutive integers is a multiple of 3:** $n+(n+1)+(n+2)=3n+3=3(n+1)$, which is a multiple of 3.
- **Sum of two odd numbers is even:** $(2a+1)+(2b+1)=2a+2b+2=2(a+b+1)$, which is even.
- $(n+1)^2-n^2$ **is odd:** $n^2+2n+1-n^2=2n+1$, which is odd.
- $(2n+1)^2-1$ **is a multiple of 8:** $4n^2+4n=4n(n+1)$. One of n and $n+1$ is even, so $n(n+1)$ is even and $4n(n+1)$ is a multiple of 8.

Examples with specific numbers are not a proof. You must show the result for all integers.

Worked example: temperature formula

Worked example 1

$F=\frac{9}{5}C+32$.

When $C=35$, $F=63+32=95$. To make C the subject: $F-32=\frac{9}{5}C$, so $C=\frac{5}{9}(F-32)$. For $F=68$: $C=\frac{5}{9}\times 36=20$.

More worked examples in exam style

Worked example 2

$T=2\pi\sqrt{\frac{l}{g}}$. **Make l the subject [3 marks, Higher].**

Divide by 2π: $\frac{T}{2\pi}=\sqrt{\frac{l}{g}}$. Square both sides: $\frac{T^2}{4\pi^2}=\frac{l}{g}$. Multiply by g: $l=\frac{gT^2}{4\pi^2}$.

Worked example 3

A taxi charges £3 plus £2.40 per mile. (a) Write a formula for the cost C of a journey of m miles. (b) Find m when $C=£21$ [4 marks].

(a) $C=3+2.4m$. (b) $21=3+2.4m$, so $18=2.4m$ and $m=7.5$ miles.

Worked example 4

Show that $\frac{x+1}{2}+\frac{x-1}{3}\equiv\frac{5x+1}{6}$ [3 marks].

Common denominator 6: $\frac{3(x+1)+2(x-1)}{6}=\frac{3x+3+2x-2}{6}=\frac{5x+1}{6}$.

Worked example 5 (Higher tier)

Prove that the difference between the squares of two consecutive odd numbers is a multiple of 8 [4 marks].

Odd numbers $2n+1$ and $2n+3$.
$(2n+3)^2-(2n+1)^2=4n^2+12n+9-4n^2-4n-1=8n+8=8(n+1)$.

Common mistakes

Watch out

- Not applying the same operation to the whole of both sides, for example dividing only part of the right-hand side.
- Making a sign error when moving a term across: $y=5-x$ gives $x=5-y$, not $x=y-5$.
- Using a numerical example as a proof.
- Forgetting the square root's order in $v^2=u^2+2as$: subtract first, then divide, then (if needed) root.

Exam technique

Exam technique

- In "prove" questions write a concluding sentence: "so this is a multiple of 3 for all integers n".
- When rearranging, write each line below the last and tick off one inverse operation per line.
- Check a rearranged formula with a simple pair of numbers.
- Do not use the same letter for different quantities in a proof: use n and m for two different integers.

Reading and using formulae

A formula is a rule that connects quantities and is written with letters. Formulae are used by engineers, nurses, builders and shopkeepers, and they have a standard layout: the subject is on its own on the left of the equals sign. When you are given a formula and

some values, you substitute and calculate. When you are asked to make a different letter the subject, you rearrange the formula so that the new letter is on its own, using inverse operations in the reverse order. The golden rule of rearranging is that whatever you do to one side of the formula you must do to the other side, and you should undo the operations in the opposite order from the usual order of operations: first undo addition and subtraction, then multiplication and division, then powers and roots.

Worked example 6: a two-step rearrangement

Make x the subject of $y = \dfrac{3x - 2}{5}$ [3 marks].

Multiply both sides by 5: $5y = 3x - 2$. Add 2 to both sides: $5y + 2 = 3x$. Divide both sides by 3: $x = \dfrac{5y + 2}{3}$. Check with numbers: take $x = 4$, which gives $y = \dfrac{10}{5} = 2$. Then $\dfrac{5 \times 2 + 2}{3} = \dfrac{12}{3} = 4$. The check confirms the rearrangement.

Worked example 7: a square root in the formula

Make g the subject of $T = 2\pi\sqrt{\dfrac{l}{g}}$ [4 marks].

Divide by 2π: $\dfrac{T}{2\pi} = \sqrt{\dfrac{l}{g}}$. Square both sides: $\dfrac{T^2}{4\pi^2} = \dfrac{l}{g}$. Multiply by g: $g \times \dfrac{T^2}{4\pi^2} = l$. Divide by $\dfrac{T^2}{4\pi^2}$: $g = \dfrac{4\pi^2 l}{T^2}$. Notice that squaring is the inverse of the square root, and that g was in the denominator so it had to be cleared first.

Worked example 8: substituting a negative value

Work out $E = \frac{1}{2}mv^2$ when $m = 4$ and $v = -3$. Then work out $F = ma - k$ when $m = 2$, $a = 5$ and $k = -7$ [3 marks].

For E: $v^2 = (-3)^2 = 9$, so $E = \frac{1}{2} \times 4 \times 9 = 18$. For F: $2 \times 5 - (-7) = 10 + 7 = 17$. Subtracting a negative gives a positive, which is where many marks are lost.

Worked example 9: forming a formula

A plumber charges £45 for a call-out plus £28 for each hour of work. Write a formula for the total cost C pounds for h hours and find how many hours are charged if the bill is £157 [3 marks].

The formula is $C = 45 + 28h$. Substitute $C = 157$: $157 = 45 + 28h$, so $28h = 112$ and $h = 4$. The plumber worked for 4 hours.

Note: showing a statement is an identity

An identity uses the symbol $\equiv$ and is true for all values of the variable. To *show* that an identity is true, work on one side only (usually the more complicated side) until you reach the other side, and write each step. Do not work on both sides at once and do not substitute a single number: one number can only show that the statement is true for that number. To *disprove* a statement one counter-example is enough.

Watch out: rearranging errors

Common errors are subtracting a term from only one side, forgetting to divide *every* term, and writing $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ when squaring. When you square both sides of $\sqrt{x} + 3 = y$ you must first isolate the root: $\sqrt{x} = y - 3$, so $x = (y-3)^2$.

Did you know?

Many of the formulae we use today were found by scientists who wrote them down as words, not with letters. The first person to use letters for known and unknown quantities was the French lawyer and mathematician François Viète in about 1591, and the equals sign $=$ was invented by the Welsh mathematician Robert Recorde in 1557 because, as he wrote, “no two things can be more equal” than two parallel lines of the same length. Before that, mathematicians wrote the word *aequalis* or its abbreviation. The formula $E = mc^2$ from Albert Einstein, with just five symbols, links energy, mass and the speed of light, and it is probably the most famous formula in history.

Otto's tip

To rearrange, undo the formula in the reverse order of how it was built. Substitute a number back in at the end to check that your new formula works.