Lesson - Factors, multiples, primes and HCF/LCM

Year: Year 10

These ideas underpin fractions, ratio, algebra and surds. On the GCSE you are expected to write numbers as a product of prime factors, use that product to find the highest common factor (HCF) and lowest common multiple (LCM), solve context problems and list outcomes systematically. Questions appear on both Foundation and Higher papers; the product rule for counting is Higher tier only.

Starter activity

Take 24 counters and arrange them in a rectangle with no gaps. How many different rectangles can you make (counting a 3×8 and an 8×3 as the same)? Write the dimensions of each. Now try 17 counters. What is special about 17? Try 36 counters and see whether you can make a square. Collect your results for the numbers 1 to 20 in a table and sort the numbers by how many rectangles they make.

Key vocabulary

Definition: factor

Factor: a whole number that divides exactly into another. The factors of 12 are 1, 2, 3, 4, 6 and 12.

Definition: multiple

Multiple: a number in the times table of another. Multiples of 7: 7, 14, 21, 28, ...

Definition: prime number

Prime number: a number with exactly two factors, 1 and itself: 2, 3, 5, 7, 11, 13, 17, 19, 23, ... Note that 1 is not prime and 2 is the only even prime.

Definition: square numbers

Square numbers: 1, 4, 9, 16, 25, ... **Cube numbers:** 1, 8, 27, 64, 125, ...

Listing factors in pairs

Method: listing factors in pairs

To list all factors of a number, work through the whole numbers from 1 and write the

factors in pairs. For 36: 1×36, 2×18, 3×12, 4×9, 6×6. The factors are 1, 2, 3, 4, 6, 9, 12, 18, 36. You can stop when the pairs meet in the middle.

Prime factorisation

Definition: product of prime factors

Writing a number as a **product of prime factors** means writing it as primes multiplied together, in index form: $84 = 2^2 \times 3 \times 7$.

Rule: prime factorisation

Every whole number greater than 1 can be written as a product of primes in exactly one way. Use a factor tree or repeated division by primes. Divide 84 by 2 to get 42, by 2 to get 21, by 3 to get 7. So $84 = 2 \times 2 \times 3 \times 7 = 2^2 \times 3 \times 7$.

Write the answer in index form and in order of size of the primes. Check by multiplying out.

HCF and LCM from prime factors

Definition: highest common factor (HCF)

The **HCF** of two numbers is the largest number that divides exactly into both.

Definition: lowest common multiple (LCM)

The **LCM** of two numbers is the smallest number that is in both times tables.

Method: HCF and LCM from prime factors

Write each number as a product of primes. Then:

- **HCF:** multiply the primes that appear in *both* numbers, using the *smallest* power of each.
- **LCM:** multiply every prime that appears in *either* number, using the *largest* power of each.

Worked example 1

$60 = 2^2 \times 3 \times 5$ **and** $84 = 2^2 \times 3 \times 7$**.**

HCF $= 2^2 \times 3 = 12$. LCM $= 2^2 \times 3 \times 5 \times 7 = 420$. A useful check is that HCF $\times$ LCM $=$ product of the two numbers: $12 \times 420 = 5040 = 60 \times 84$.

Using HCF and LCM in problems

Key fact: when to use HCF and when to use LCM

Use **LCM** when events repeat and you want to know when they happen together (buses, lights, bells). Use **HCF** when you want to split something into the largest equal groups or cut the largest identical squares from a rectangle.

Worked example 2

One bus leaves every 12 minutes and another every 15 minutes.

Both leave at 08:00. When do they next leave together? The LCM of 12 and 15 is 60, so at 09:00.

Worked example 3

A rectangle 84 cm by 126 cm is cut into identical squares with no waste.

Find the largest possible square. $84 = 2^2 \times 3 \times 7$, $126 = 2 \times 3^2 \times 7$. HCF $= 2 \times 3 \times 7 = 42$, so squares of side 42 cm.

Squares, cubes and roots from prime factors

Key fact: squares from prime factors

A number is a perfect square when every prime has an even power. $144 = 2^4 \times 3^2$, so $\sqrt{144} = 2^2 \times 3 = 12$. To make 72 a square, notice $72 = 2^3 \times 3^2$: the power of 2 is odd, so multiply by 2 to get 144.

Systematic listing

Key fact: systematic listing

When asked to find all possible outcomes, list them in an organised way, for example in alphabetical or numerical order, so that none is missed or repeated. How many two-digit numbers can be made from the digits 1, 2 and 3 with no digit repeated? List: 12, 13, 21, 23, 31, 32. There are 6.

The product rule for counting (Higher tier)

Rule: the product rule for counting (Higher tier)

If there are a ways to do one thing and b ways to do another, there are $a \times b$ ways to do both. With 4 shirts, 3 pairs of trousers and 2 hats there are $4 \times 3 \times 2 = 24$ outfits. If

items cannot repeat, the number of choices goes down each time: a 4-digit PIN with no repeated digit has $10 \times 9 \times 8 \times 7 = 5040$ possibilities.

More worked examples in exam style

Worked example 4

Find the HCF and LCM of 48 and 180 [4 marks].

$48 = 2^4 \times 3$ and $180 = 2^2 \times 3^2 \times 5$. HCF: smallest powers of common primes, $2^2 \times 3 = 12$. LCM: largest powers of every prime, $2^4 \times 3^2 \times 5 = 720$. Check: $12 \times 720 = 8640 = 48 \times 180$.

Worked example 5

Dev says "$2 \times 3^2 \times 5$ is a multiple of 15". Is he right? [2 marks].

$15 = 3 \times 5$, and the product contains both 3 and 5, so it is a multiple of 15. Yes. In fact $2 \times 9 \times 5 = 90 = 15 \times 6$.

Worked example 6

Two cogs have 24 and 40 teeth. They start with the marked teeth touching. After how many turns of the smaller cog are the marks next touching? [3 marks].

LCM of 24 and 40: $24 = 2^3 \times 3$, $40 = 2^3 \times 5$; LCM $= 2^3 \times 3 \times 5 = 120$ teeth. Turns of the smaller cog: $120 \div 24 = 5$.

Worked example 7

A teacher has 60 pencils and 84 rulers and wants identical packs using all items. What is the largest number of packs and what is in each? [3 marks].

HCF of 60 and 84 is 12, so 12 packs, each with 5 pencils and 7 rulers.

Common mistakes

Watch out

- Saying 1 is a prime number, or forgetting that 2 is prime.
- Mixing up HCF (smallest powers, common primes only) and LCM (largest powers, all primes).
- Stopping a factor tree too early, with a composite number left at the end of a branch.
- Listing the factors of a number but missing the pair partner, for example forgetting 18 when listing the factors of 36.

Exam technique

Exam technique

- "Write as a product of prime factors" always needs the answer in the form $2^a \times 3^b \times \ldots$; a factor tree alone does not earn the final mark.
- When a problem says "at the same time", "again" or "together", think LCM. When it says "largest", "equal groups" or "no waste", think HCF.
- For "list all", write the outcomes in order and count them at the end.
- Use the check HCF $\times$ LCM = product when you can; it catches arithmetic slips quickly.

Why primes matter

Every whole number greater than 1 can be built from prime numbers by multiplication, and in only one way, apart from the order of the factors. This is called the fundamental theorem of arithmetic and it is the reason why prime factorisation is so useful. Prime factors act like the DNA of a number: once you know them you can read off its factors, test whether it is a square or a cube, and find the HCF and LCM of two numbers quickly. To factorise a number, divide repeatedly by the smallest prime that goes in exactly, starting with 2, then 3, 5, 7, and so on. You can stop when the number left is 1, and you only need to test primes up to the square root of the number to decide whether it is prime.

Worked example 8: testing for a prime

Is 223 a prime number? [2 marks]

The square root of 223 is about 14.9, so we only need to test the primes 2, 3, 5, 7, 11 and 13. The number is odd, so 2 does not divide it. The digit sum is $2 + 2 + 3 = 7$, so 3 does not divide it. It does not end in 0 or 5. $223 \div 7 = 31.86$, $223 \div 11 = 20.27$ and $223 \div 13 = 17.15$, none of them whole numbers. So 223 is prime.

Worked example 9: the number of factors

How many factors has 600? [3 marks]

Write 600 in prime factors: $600 = 2^3 \times 3 \times 5^2$. Add 1 to each index and multiply: $(3 + 1)(1 + 1)(2 + 1) = 4 \times 2 \times 3 = 24$. So 600 has 24 factors. This works because each factor is made by choosing how many 2s (0 to 3), how many 3s (0 or 1) and how many 5s (0 to 2), which gives $4 \times 2 \times 3$ choices.

Worked example 10: HCF and LCM with a diagram

Find the HCF and LCM of 72 and 108 using prime factors [4 marks].

$72 = 2^3 \times 3^2$ and $108 = 2^2 \times 3^3$. For the HCF take the *lower* power of each prime: $2^2 \times 3^2 = 36$. For the LCM take the *higher* power of each prime: $2^3 \times 3^3 = 216$. Check: $36 \times 216 = 7776$ and $72 \times 108 = 7776$, so the product of the two numbers equals the

product of the HCF and LCM.

Worked example 11: a reasoning question

Mia says, “If a number is divisible by 6, it is divisible by 12.” Show that she is wrong [2 marks].

A counter-example is enough. Take 18: $18 \div 6 = 3$, so it is divisible by 6, but $18 \div 12 = 1.5$, so it is not divisible by 12. One counter-example proves the statement false. (The statement would be right the other way round: a number divisible by 12 is always divisible by 6.)

Note: divisibility tests

A number is divisible by 2 if it ends in an even digit, by 3 if its digit sum is divisible by 3, by 4 if its last two digits form a number divisible by 4, by 5 if it ends in 0 or 5, by 6 if it passes the tests for 2 and 3, by 9 if its digit sum is divisible by 9, and by 10 if it ends in 0. These tests are quick ways of finding the first prime factors of a large number in a non-calculator paper.

Watch out: HCF or LCM?

If a question is about sharing or cutting into equal groups or pieces, think HCF, because the answer is a number that goes *into* both. If a question is about events repeating at the same time, think LCM, because the answer is a number that both *go into*. The HCF is never bigger than the smaller number, and the LCM is never smaller than the larger number.

Did you know?

The ancient Greeks, including Euclid, knew that there are infinitely many prime numbers. The proof is a famous one-page argument: suppose there were only a finite list of primes, multiply them all together and add 1, and the new number has a prime factor that is not on the list, which is a contradiction. The largest known prime numbers are found by computers: in 2018 the number $2^{82589933} - 1$ was discovered, which has nearly 25 million digits. Large primes are not just curiosities: they protect your online banking and messages, because multiplying two big primes is easy but splitting the product back into primes is extremely hard.

Otto's tip

Line up the prime factors of the two numbers in a table. For the HCF, circle what is in both columns. For the LCM, take everything. Then multiply out only at the end.