Lesson - Area, volume, surface area, arcs and sectors

Year: Year 10

Mensuration questions test whether you can choose and use the right formula, keep units consistent and show clear working. Area, prisms, cylinders, arcs and sectors are on both GCSE tiers. Volume and surface area of cones, spheres and pyramids, and composite solids, are in the Higher tier content; they are marked (Higher) here.

Starter activity

Place a cuboid-shaped box (for example a cereal packet) and a cylindrical tin on the table. Without a ruler, which do you think holds more? Now measure the box's length, width and height and the tin's radius and height, and work out both volumes. Then wrap a sheet of paper round the tin to see its curved surface open out into a rectangle. What are the sides of that rectangle, and how do they connect to the radius and height?

Area of 2D shapes

Definition: area and perimeter

Area is the space inside a 2D shape (measured in square units). **Perimeter** is the distance around it (measured in units of length).

Key fact: area formulae

Shape	Area
Rectangle	$l \times w$
Triangle	$\frac{1}{2} \times \text{base} \times \text{perpendicular height}$
Parallelogram	base $\times$ perpendicular height
Trapezium	$\frac{1}{2}(a+b)h$
Circle	πr^2

Triangle with base 8 and height 5: $\frac{1}{2} \times 8 \times 5 = 20$. Trapezium with parallel sides 6 and 10 and height 4: $\frac{1}{2} \times 16 \times 4 = 32$. Circle of radius 7: $\pi \times 49 = 153.9$, or 154 to 3 s.f. Circumference $= \pi d = 2\pi r$: diameter 12 gives $12\pi = 37.7$. Always use the *perpendicular* height. For a compound shape, split into rectangles and triangles or subtract a missing piece: a 10×6 rectangle with a 4×3 corner removed has area $60 - 12 = 48$.

Volume and surface area of prisms

Definition: prism and surface area

A **prism** has the same cross-section all the way along its length. Its **surface area** is the total area of all its faces.

Rule: volume and surface area of prisms

A **prism** has the same cross-section throughout.
$\text{Volume} = \text{area of cross-section} \times \text{length}$. A triangular prism with cross-section area 12 and length 10 has volume 120. A cuboid $5 \times 4 \times 3$ has volume 60 and surface area $2(20 + 15 + 12) = 94$. Surface area is the total area of all the faces, so draw the net.

Cylinders

Rule: cylinders

$V = \pi r^2 h$ and total surface area $= 2\pi rh + 2\pi r^2$ (curved surface plus two circles). For $r = 3$, $h = 8$: $V = 72\pi = 226$ cm^3; surface area $= 48\pi + 18\pi = 66\pi = 207$ cm^2. Leave answers in terms of π when asked.

Units

Key fact: units of length, area and volume

Convert to the same units before calculating. Area units are squared and volume units cubed. A cube of side 4 cm has volume 64 cm$^3 = 64\,000$ mm^3 (because 1 cm $= 10$ mm, and 4 cm $= 40$ mm, $40^3 = 64\,000$).

Arcs and sectors

Rule: arcs and sectors

A sector with angle θ is the fraction $\dfrac{\theta}{360}$ of the whole circle.

- Arc length $= \dfrac{\theta}{360} \times 2\pi r$.
- Sector area $= \dfrac{\theta}{360} \times \pi r^2$.

Radius 6 and angle $60°$: arc $= \frac{1}{6} \times 12\pi = 2\pi = 6.28$ cm; area $= \frac{1}{6} \times 36\pi = 6\pi = 18.8$ cm^2. Angle $120°$ and radius 9: area $= \frac{1}{3} \times 81\pi = 27\pi$. The **perimeter** of a sector is the arc plus *two radii*: radius 5, angle $72°$: arc $= 2\pi = 6.28$, perimeter $= 10 + 6.28 = 16.3$

cm. A semicircle of radius 10 has perimeter $10\pi + 20 = 51.4$.

Method: arc length and sector area

1. Write the fraction of the circle: $\dfrac{\theta}{360}$.
2. Arc length: multiply the fraction by $2\pi r$.
3. Sector area: multiply the fraction by πr^2.
4. Perimeter of a sector: arc length plus two radii.

Cones, spheres and pyramids (Higher tier)

Rule: cones, spheres and pyramids (Higher tier)

- **Cone:** $V = \frac{1}{3}\pi r^2 h$; curved surface area $= \pi r l$, where l is the slant height, and $l^2 = r^2 + h^2$.
- **Sphere:** $V = \frac{4}{3}\pi r^3$; surface area $= 4\pi r^2$.
- **Pyramid:** $V = \frac{1}{3} \times \text{base area} \times \text{height}$.

Cone $r = 5$, $h = 12$: $l = \sqrt{25 + 144} = 13$; $V = \frac{1}{3}\pi \times 25 \times 12 = 100\pi = 314$ cm^3; curved area $= \pi \times 5 \times 13 = 65\pi = 204$ cm^2. Sphere $r = 6$: $V = \frac{4}{3}\pi \times 216 = 288\pi = 905$ cm^3; surface area $144\pi = 452$ cm^2. Square-based pyramid with base 8 and height 9: $V = \frac{1}{3} \times 64 \times 9 = 192$ cm^3. Hemisphere $r = 4$: $V = \frac{2}{3}\pi \times 64 = \frac{128\pi}{3} = 134$ cm^3; total surface area $2\pi r^2 + \pi r^2 = 3\pi r^2 = 48\pi = 151$ cm^2.

More worked examples in exam style

Worked example 1

A cylinder of radius 4 cm has volume 200 cm^3. Find its height to 3 s.f. [3 marks].

$\pi \times 16 \times h = 200$, so $h = \dfrac{200}{16\pi} = 3.98$ cm.

Worked example 2

A cylindrical tank has radius 0.5 m and height 1.2 m. How many litres does it hold? [3 marks].

$V = \pi \times 0.25 \times 1.2 = 0.942$ m^3. 1 m$^3 = 1000$ litres, so 942 litres.

Worked example 3

A running track is a rectangle 80 m by 70 m with a semicircle (diameter 70 m) on each short end.

Find the perimeter. The two semicircles make a circle of diameter 70, circumference $70\pi = 220$ m. Plus two straights of 80 m: total $220 + 160 = 380$ m.

Worked example 4 (Higher tier)

A metal sphere of radius 3 cm is melted and recast into a cone of radius 3 cm. Find the height of the cone [4 marks].

Sphere volume 36π. Cone volume $\frac{1}{3}\pi \times 9 \times h = 3\pi h$. $3\pi h = 36\pi$, so $h = 12$ cm.

Common mistakes

Watch out

- Using the slant height as the perpendicular height of a triangle.
- Using the diameter instead of the radius in πr^2.
- Forgetting the two radii when finding the perimeter of a sector.
- Forgetting the $\frac{1}{3}$ for cones and pyramids.
- Mixing units, such as cm and m, in one calculation.

Exam technique

Exam technique

- Write the formula before substituting.
- Keep the full calculator value until the last line, then round.
- Include units: cm, cm^2 or cm^3 show whether you found a length, an area or a volume.

Units and measurements

Area and volume problems are full of units, and a large number of marks are lost by mixing them. Length uses units such as mm, cm, m and km. Area uses square units because it measures how many squares of side 1 unit fit inside a shape, so $1\ \text{m}^2 = 100\ \text{cm} \times 100\ \text{cm} = 10\,000\ \text{cm}^2$. Volume uses cubic units, so $1\ \text{m}^3 = 1\,000\,000\ \text{cm}^3$. Capacity is measured in litres where $1\ \text{litre} = 1000\ \text{cm}^3$. Before you start a question, write every length in the same unit, then convert at the end if you have to. A common slip is to convert lengths with the factor 100 and then forget to square it for an area.

Worked example 5: a compound shape

A shape is a rectangle 10 cm by 6 cm with a semicircle of diameter 6 cm

added on one short side. Find its area and perimeter to 1 decimal place [5 marks].

Rectangle area: $10 \times 6 = 60$ cm². Semicircle radius: 3 cm, so its area is $\frac{1}{2}\pi \times 3^2 = 4.5\pi = 14.14$ cm². Total area $= 60 + 14.14 = 74.1$ cm². For the perimeter, include the two long sides, the opposite short side and the curved edge, but not the 6 cm side where the semicircle joins. Curved edge $= \frac{1}{2} \times \pi \times 6 = 3\pi = 9.42$ cm. Perimeter $= 10 + 10 + 6 + 9.42 = 35.4$ cm.

Worked example 6: finding an angle

A sector of radius 9 cm has arc length 12 cm. Find its angle to the nearest degree [3 marks].

Arc length is $\frac{\theta}{360} \times 2\pi r$. Substitute: $12 = \frac{\theta}{360} \times 2\pi \times 9 = \frac{\theta}{360} \times 56.55$. So $\theta = \frac{12 \times 360}{56.55} = 76.4$, which is 76° to the nearest degree. To check, find the arc of a 76° sector: $\frac{76}{360} \times 56.55 = 11.9$ cm. This is close to 12 cm.

Worked example 7: surface area of a cylinder

A cylinder has radius 4 cm and height 9 cm. Find its total surface area in terms of π [3 marks].

The two circular ends have area $2 \times \pi \times 4^2 = 32\pi$. The curved surface opens out to a rectangle with width equal to the circumference $2\pi \times 4 = 8\pi$ and height 9, so its area is $8\pi \times 9 = 72\pi$. Total $= 32\pi + 72\pi = 104\pi$ cm², which is 326.7 cm² to 1 decimal place. Check with the formula $2\pi r(r + h) = 2\pi \times 4 \times 13 = 104\pi$.

Worked example 8: volume in litres

A fish tank is a cuboid 60 cm long, 30 cm wide and 35 cm high. It is filled to 4 cm below the top. How many litres of water does it hold? [3 marks]

Depth of water $= 35 - 4 = 31$ cm. Volume $= 60 \times 30 \times 31 = 55\,800$ cm³. Convert: $55\,800 \div 1000 = 55.8$ litres. If you instead found the volume of the full tank, $60 \times 30 \times 35 = 63\,000$ cm³ or 63 litres, you would lose the marks for using the correct depth.

Note: formulae to learn

You are expected to know area formulae for rectangles, triangles, parallelograms, trapeziums and circles, the volume of a prism (area of cross-section times length) and of a cylinder ($\pi r^2 h$), and the circumference of a circle ($2\pi r$). For Higher tier the formulae for the cone, sphere and pyramid are not given on the formula sheet in every paper, so check what your exam board supplies.

Watch out: height and slant height

The area of a triangle or a cone’s volume uses the *perpendicular* height. The slope of the side is longer. For a cone the curved surface area uses the slant height l and the volume uses the vertical height h, linked by $l^2 = r^2 + h^2$.

Did you know?

The number pi has fascinated mathematicians for almost 4000 years. The ancient Babylonians used the value 3.125 and the Egyptians about 3.16. Around 250 BC Archimedes of Syracuse found that pi lies between $\frac{223}{71}$ and $\frac{22}{7}$ by trapping a circle between two polygons with 96 sides. In 2022 computers calculated pi to more than 100 trillion decimal places, but in practice engineers who build bridges and rockets need fewer than 20. Archimedes was so proud of his discovery that a sphere and cylinder were carved on his tomb, to show that the volume of a sphere is two thirds of the cylinder around it.

Otto's tip

Sketch the shape and label every length with its units. Write the formula first, then substitute. Put your final answer in the form requested, either an exact multiple of pi or a rounded decimal with units.