Lesson - Algebraic notation, expanding and factorising

Year: Year 10

Algebra is the language of GCSE maths. This chapter covers writing algebra correctly, substituting, collecting like terms, expanding brackets and factorising. It leads straight into equations, quadratics and proof. Factorising $ax^2 + bx + c$ with $a \neq 1$ and algebraic fractions are Higher tier only.

Starter activity

Imagine you buy 3 packets of crisps at c pence each and 2 bottles of water at w pence each. Write an expression for the total cost. Now your friend buys 5 packets of crisps and 4 bottles. Write their cost and add the two bills. Which terms could you combine? Which could you not combine, and why? Try again with $c = 80$ and $w = 65$ to check that your simplified expression gives the same total.

Notation and vocabulary

Definition: algebraic notation and vocabulary

- $3a$ means $3 \times a$; ab means $a \times b$; a^2 means $a \times a$; $\frac{a}{b}$ means $a \div b$.
- A **term** is a number, letter or product, such as $5x$ or $-3y^2$. An **expression** has no equals sign, an **equation** has one, a **formula** links variables, and an **identity** is true for all values (written $\equiv$).
- Write the number first and letters in alphabetical order: $5ab$, not $b5a$.

Substitution

Rule: substitution

Replace each letter by its value, using brackets for negative numbers. If $x = -2$ then $3x^2 - 4x + 1 = 3(-2)^2 - 4(-2) + 1 = 12 + 8 + 1 = 21$. Remember that $(-2)^2 = 4$, not -4.

Simplifying

Definition: like terms

Like terms have the same letters with the same powers, such as $3a$ and $-5a$, or $2x^2$ and $7x^2$. Only like terms can be collected.

Key fact: collecting like terms

Collect like terms (same letters with the same powers). $3a + 2b - a + 5b = 2a + 7b$. For products multiply the numbers then the letters: $4x \times 3y = 12xy$. For quotients cancel common factors: $\frac{6x^2y}{2xy} = 3x$.

Expanding brackets

Rule: expanding brackets

Multiply every term inside the bracket by the term outside: $3(2x - 5) = 6x - 15$. With two brackets multiply each term in the first by each term in the second (FOIL or a grid) and collect like terms.

$(x + 4)(x + 5) = x^2 + 5x + 4x + 20 = x^2 + 9x + 20$
$(x - 3)(x + 7) = x^2 + 7x - 3x - 21 = x^2 + 4x - 21$
$(2x + 1)(x - 6) = 2x^2 - 12x + x - 6 = 2x^2 - 11x - 6$

Squaring a bracket means writing it twice: $(x - 5)^2 = (x - 5)(x - 5) = x^2 - 10x + 25$. It is not $x^2 + 25$.

Method: expanding double brackets

1. Multiply every term in the first bracket by every term in the second (FOIL or a grid).
2. Write out all four products.
3. Collect the like terms.

Factorising

Definition: factorising

Factorising is expanding in reverse. Always look for a common factor first and take out the highest one: $12x + 18 = 6(2x + 3)$ and $5x^2 - 15x = 5x(x - 3)$.

Factorising $x^2 + bx + c$

Method: factorising x squared plus bx plus c

Find two numbers that multiply to c and add to b.

- $x^2 + 7x + 12$: numbers 3 and 4, so $(x + 3)(x + 4)$.
- $x^2 - 5x - 14$: numbers -7 and 2 (product -14, sum -5), so $(x - 7)(x + 2)$.

Check by expanding: the middle term and the last term should match.

Difference of two squares

Rule: the difference of two squares

$a^2 - b^2 = (a - b)(a + b)$. So $x^2 - 49 = (x - 7)(x + 7)$ and $9y^2 - 16 = (3y - 4)(3y + 4)$. There must be a minus between two perfect squares.

Factorising $ax^2 + bx + c$ (Higher tier)

Method: factorising when a is not 1 (Higher tier)

Multiply $a \times c$, find two numbers that multiply to ac and add to b, split the middle term and factorise in pairs. For $2x^2 + 7x + 3$: $ac = 6$, numbers 1 and 6.
$2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = (2x + 1)(x + 3)$.

Algebraic fractions (Higher tier)

Rule: simplifying algebraic fractions (Higher tier)

Factorise the top and bottom completely, then cancel common *factors* (never common terms). $\dfrac{x^2 - 9}{x^2 + 5x + 6} = \dfrac{(x - 3)(x + 3)}{(x + 2)(x + 3)} = \dfrac{x - 3}{x + 2}$.

Worked example in exam style

Worked example 1

Show that $(n + 3)^2 - 9 = n(n + 6)$ [2 marks].

Expand: $n^2 + 6n + 9 - 9 = n^2 + 6n = n(n + 6)$. Each line is shown, ending with the form asked for.

More worked examples

Worked example 2

Expand and simplify $(3x - 2)(x + 4) - 5x$ [3 marks].

$3x^2 + 12x - 2x - 8 - 5x = 3x^2 + 5x - 8$. Notice that the $-5x$ is collected only after the brackets are expanded.

Worked example 3

Factorise fully $8x^2y - 12xy^2$ [2 marks].

The highest common factor of 8 and 12 is 4, and both terms contain x and y. $4xy(2x - 3y)$. Check: $4xy \times 2x = 8x^2y$ and $4xy \times 3y = 12xy^2$.

Worked example 4

Factorise $x^2 - 6x + 9$ [2 marks].

Numbers with product 9 and sum -6: -3 and -3. $(x - 3)^2$. This is a perfect square, so the two brackets are the same.

Worked example 5

A square has side $(2x + 1)$ cm. Show that its area is $4x^2 + 4x + 1$ [2 marks].

$(2x + 1)^2 = 4x^2 + 2x + 2x + 1 = 4x^2 + 4x + 1$.

Worked example 6 (Higher tier)

Simplify $\dfrac{3x^2 + x - 2}{3x^2 - 5x + 2}$ [4 marks].

Top: $ac = -6$, numbers 3 and -2: $3x^2 + 3x - 2x - 2 = 3x(x + 1) - 2(x + 1) = (3x - 2)(x + 1)$. Bottom: $ac = 6$, numbers -3 and -2: $3x^2 - 3x - 2x + 2 = 3x(x - 1) - 2(x - 1) = (3x - 2)(x - 1)$. Cancel $(3x - 2)$: $\dfrac{x + 1}{x - 1}$.

Common mistakes

Watch out

- Writing $(x + 3)^2 = x^2 + 9$: the middle term $6x$ is missing.
- Sign slips when expanding a negative: $-2(x - 1) = -2x + 2$, not $-2x - 2$.
- Not taking out the highest common factor: $6x^2 + 9x$ is $3x(2x + 3)$, not $x(6x + 9)$.
- Cancelling terms instead of factors in a fraction, such as cancelling the x in $\frac{x+3}{x}$.

Exam technique

Exam technique

- Write each expansion on its own line and circle the like terms before collecting them.

- Factorise "fully" means no further common factor is left: check each bracket.
- Substitution questions often use negative values: write the brackets in.
- If a question says "show that", every line must be visible and the final line must match the target exactly.

Building and checking expressions

Algebra works like a language with its own grammar. A formula or expression is built from letters that stand for numbers, so every rule of arithmetic carries over. The three habits that make algebra reliable are writing each step on its own line, using brackets when you substitute negatives, and checking your result with a simple number. For example, to check that $(x+3)(x+4) = x^2 + 7x + 12$, try $x = 2$: the left side is $5 \times 6 = 30$ and the right side is $4 + 14 + 12 = 30$. A match does not prove the identity, but a mismatch proves that you made an error.

Worked example 7: substituting into a formula

Use $v = u + at$ to find v when $u = -4$, $a = 3$ and $t = 5$ (non-calculator).

Replace each letter with its value, using brackets for the negative: $v = (-4) + 3 \times 5$. Multiply first: $3 \times 5 = 15$. Then add: $-4 + 15 = 11$. So $v = 11$. If you wrote $-4 + 3 \times 5$ as $(-4 + 3) \times 5$ you would get -5, which shows why the order of operations (BIDMAS) matters.

Worked example 8: expanding with a negative in front

Expand and simplify $3(x+2) - 2(x-5)$ [3 marks].

Expand the first bracket: $3x + 6$. Expand the second bracket: $-2 \times x = -2x$ and $-2 \times (-5) = +10$. Now collect like terms: $3x + 6 - 2x + 10 = x + 16$. The common slip is to write $-2x - 10$; the minus sign in front of the bracket changes the sign of *every* term inside it.

Worked example 9: the square of a bracket

Expand $(3x-2)^2$ and then check with $x = 1$.

Write it as two brackets: $(3x-2)(3x-2)$. Multiply: $9x^2 - 6x - 6x + 4 = 9x^2 - 12x + 4$. Check with $x = 1$: left side $(3-2)^2 = 1$; right side $9 - 12 + 4 = 1$. Good. The wrong answer $9x^2 + 4$ would give 13, which does not match.

Worked example 10: factorising in two stages

Factorise fully $2x^2 - 18$ [3 marks].

First take out the common factor 2: $2(x^2 - 9)$. Now $x^2 - 9$ is a difference of two squares, $x^2 - 3^2$, so it factorises as $(x-3)(x+3)$. The full answer is $2(x-3)(x+3)$. Many students stop after the first stage, but a question that says “fully” expects both.

Worked example 11: forming and simplifying in context

A cinema charges £a for an adult and £c for a child. A family buys 2 adult and 3 child tickets and a group buys 5 adult and 4 child tickets. Write the total cost of both groups and simplify.

Family: $2a + 3c$. Group: $5a + 4c$. Total: $2a + 3c + 5a + 4c = 7a + 7c = 7(a + c)$. The factorised form tells us that the combined bill equals the cost of 7 pairs of one adult and one child ticket.

Note: expanding a longer bracket

Whether you use the grid method, the “smile and eyebrows” method or FOIL, all the products must appear before you collect like terms. A grid is the safest method for longer brackets such as $(x + 2)(x^2 - 3x + 4)$, because each term in the first bracket multiplies each term in the second: $x^3 - 3x^2 + 4x + 2x^2 - 6x + 8 = x^3 - x^2 - 2x + 8$. Count your products: a 2-term bracket times a 3-term bracket must give $2 \times 3 = 6$ products.

Choosing the right tool

Many GCSE questions in this topic use the command words *simplify*, *expand*, *factorise* and *show that*, and each one asks for a different action. To simplify you collect like terms and the expression gets shorter. To expand you multiply out brackets and the expression usually gets longer. To factorise you do the reverse of expanding and put brackets back in. To show that you start from one side and reach the other, writing every line, without using the answer you are trying to prove. If you are unsure which action is needed, look at the shape of the expression: brackets with nothing outside need expanding, a long sum with a common factor needs factorising, and a sum of several terms with repeated letters needs simplifying. Practising the four actions in mixed order, rather than one at a time, is the best way to prepare for a real exam paper.

Watch out: signs and squares

Two sign errors cost marks more than any others. First, a minus sign in front of a bracket must be applied to every term, so $-(2x - 3) = -2x + 3$. Second, $(-x)^2 = x^2$ but $-x^2$ means $-(x^2)$, so with $x = 4$ the first is 16 and the second is -16. Write brackets round negative numbers on your calculator too.

Did you know?

The word algebra comes from the Arabic *al-jabr*, from a book written around the year 820 by the mathematician Muhammad ibn Musa al-Khwarizmi in Baghdad. “Al-jabr” means “restoring” or “completing”, which describes the step of moving a subtracted term to the other side. Our word algorithm also comes from his name. The use of letters for unknown numbers came much later: the French mathematician François Viète used letters in the 1590s, and René Descartes introduced the habit of using x, y

and z for unknowns and a, b and c for known values. Every time you write $3x + 4$ you are using ideas that took more than 800 years to develop.

Otto's tip

Always check a factorisation by expanding it. If it does not give back the original, change the signs of your two numbers and try again.