Test answers - Vectors and systems of linear equations

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. $\mathbf{a}\cdot\mathbf{b} = 3+0+8 = 11$, $|\mathbf{a}| = \sqrt{1+4+4} = 3$ and $|\mathbf{b}| = \sqrt{9+0+16} = 5$ (1). So $\cos\theta = \frac{11}{15}$ and $\theta \approx 42.8^\circ$ (1).

b. $\mathbf{a}\times\mathbf{b} = (2\cdot 4 - 2\cdot 0,\ 2\cdot 3 - 1\cdot 4,\ 1\cdot 0 - 2\cdot 3) = (8, 2, -6)$ (1). The triangle has half the area of the parallelogram: $\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = \frac{1}{2}\sqrt{64+4+36} = \frac{1}{2}\sqrt{104} \approx 5.10$ square units (1).

In (a), the formula used is $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$; a positive dot product means the angle is acute. In (b), check that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both vectors: $(8,2,-6)\cdot(1,2,2) = 8+4-12 = 0$ and $(8,2,-6)\cdot(3,0,4) = 24-24 = 0$.

Answer to Question 2 (4 marks)

Two vectors in the plane are  and . Their cross product gives a normal:  (2). The plane is $6x + 3y + 2z = d$, and substituting A gives $d = 6$, so the plane is $6x + 3y + 2z = 6$ (1). Check: B gives 6 and C gives 6. The distance from the origin is $\dfrac{|0-6|}{\sqrt{36+9+4}} = \dfrac{6}{\sqrt{49}} = \dfrac{6}{7}$ (1).

The intercept form $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$ gives the same plane after multiplying by 6. The normal $(6, 3, 2)$ has length exactly 7.

Answer to Question 3 (4 marks)

Use row 1 to clear the first column: $R_2 - 2R_1$ gives $(0, 1, -1 \mid -3)$ and $R_3 - R_1$ gives $(0, -2, 1 \mid 4)$ (1). Then $R_3 + 2R_2$ gives $(0, 0, -1 \mid -2)$, so the matrix is in echelon form (1). Back substitution: $-z = -2$ gives $z = 2$; $y - z = -3$ gives $y = -1$; $x + y + z = 2$ gives $x = 1$ (2). Check in the third equation: $1 + 1 + 4 = 6$.

The system has a unique solution because elimination produced a pivot in every column. Check the second equation as well: $2 - 3 + 2 = 1$.

Answer to Question 4 (4 marks)

a. The coefficient matrix $\begin{pmatrix} 1 & 2 \\ 2 & k \end{pmatrix}$ has determinant $k - 4$. There is exactly one solution when it is non-zero, that is $k \neq 4$ (1).
b. When $k = 4$, the left-hand side of the second equation is $2x + 4y = 2(x + 2y)$, so it reads $2(x + 2y) = c$ (1). If $c = 6$, it is the first equation multiplied by 2: the same line, so there are infinitely many solutions (1). If $c \neq 6$, the lines are parallel and distinct, so there is no solution (1).

Answer to Question 5 (4 marks)

a. A point of the beam is $(2 + t,\ 1 + t,\ t)$. Substituting into the plane: $2 + t + 2(1 + t) + t = 4 + 4t = 10$, so $t = 1.5$ (1). The point of impact is $(3.5,\ 2.5,\ 1.5)$ (1). Check: $3.5 + 5 + 1.5 = 10$.
b. The angle between a line and a plane is the complement of the angle between the line and the normal $\mathbf{n} = (1, 2, 1)$. So $\sin\varphi = \dfrac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \dfrac{4}{\sqrt{3}\sqrt{6}}$ (1), which gives $\varphi \approx 70.5^\circ$ (1).