Lesson - Vectors and systems of linear equations

Year: University Year 1

Vectors describe direction and magnitude, and systems of linear equations are solved by systematic row operations. Together they are the starting point of linear algebra.

Starter activity

A builder needs to know whether three steel beams, each described by a straight line in space, actually meet at one point, and a surveyor wants the height of a sloping roof above a given spot. Both questions come down to equations of lines and planes. Before reading on, write down the equation of the floor $z = 0$ and of a wall, and ask yourself: how many equations do you need to pin down a single point in space?

Vectors in R^n

Definition: dot product and length

A **vector** in $\mathbb{R}^3$ is an ordered triple $\mathbf{a} = (a_1, a_2, a_3)$, added and scaled componentwise. The **length** is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. The **dot product** is $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$.

Rule: dot product results

$\mathbf{a} \perp \mathbf{b}$ exactly when $\mathbf{a} \cdot \mathbf{b} = 0$. The **Cauchy-Schwarz inequality** $|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}||\mathbf{b}|$ holds for all vectors. The **projection** of $\mathbf{a}$ onto $\mathbf{b}$ is $\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$.

Example 1

Find the angle between $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, -1, 4)$.

$\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 12 = 12$, $|\mathbf{a}| = \sqrt{14}$, $|\mathbf{b}| = \sqrt{21}$. So $\cos\theta = \frac{12}{\sqrt{294}} \approx 0.700$ and $\theta \approx 45.6^\circ$.

The cross product

Definition: cross product

In $\mathbb{R}^3$, $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$. It is perpendicular to both vectors, and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ is the area of the parallelogram they span.

Property: cross product facts

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ if and only if the vectors are parallel. The scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ gives the signed volume of a parallelepiped.

Example 2

Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (2, -1, 4)$.

$\mathbf{a} \times \mathbf{b} = (8 + 3,\ 6 - 4,\ -1 - 4) = (11, 2, -5)$ and $|\mathbf{a} \times \mathbf{b}| = \sqrt{150} = 5\sqrt{6}$.

Lines and planes

Rule: equations of lines and planes

A line through $\mathbf{p}$ with direction $\mathbf{d}$ is $\mathbf{r} = \mathbf{p} + t\mathbf{d}$. A plane with normal $\mathbf{n}$ through $\mathbf{p}$ is $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{p}$. The distance from the origin to $\mathbf{n} \cdot \mathbf{r} = d$ is $\frac{|d|}{|\mathbf{n}|}$. For the plane through three points, use the cross product of two edge vectors as the normal.

Example 3

Find the plane with normal $\mathbf{n} = (2, -1, 4)$ through $(1, 2, 3)$ and its distance from the origin.

$2x - y + 4z = 2 - 2 + 12 = 12$. The distance is $\frac{12}{\sqrt{21}} \approx 2.62$.

Systems of linear equations

Definition: augmented matrix and row operations

A system of m linear equations in n unknowns can be written $A\mathbf{x} = \mathbf{b}$, with augmented matrix $[A \mid \mathbf{b}]$. The **elementary row operations** are: swap two rows; multiply a row by a non-zero constant; add a multiple of one row to another. They do not change the solution set.

Method: Gaussian elimination

1. Write the augmented matrix.
2. Use row operations to make zeros below each leading entry (row echelon form).
3. Back-substitute from the bottom row up, or continue to reduced row echelon form (leading 1s, zeros above and below) to read off the answer.
4. Substitute the solution into every original equation.

Example 4

Solve $x+y+z=6$, $2x-y+z=3$, $x+2y-z=2$.

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ give $-3y-z=-9$ and $y-2z=-4$. From the second, $y=2z-4$; substituting, $-6z+12-z=-9$, so $z=3$, $y=2$, $x=1$. The unique solution is $(1,2,3)$.

How many solutions?

Rule: number of solutions

A system has either no solution, exactly one, or infinitely many. The **rank** of A is the number of non-zero rows in echelon form. The system is *inconsistent* if the echelon form contains a row $[0\,0\,\cdots\,0 \mid c]$ with $c \neq 0$. If it is consistent and rank equals the number of unknowns, the solution is unique; if rank is less, there are $n - \text{rank}$ free parameters.

Property: homogeneous systems

A **homogeneous** system $A\mathbf{x}=\mathbf{0}$ always has the trivial solution, and has non-trivial solutions if and only if rank .

Example 5

Discuss $x+y=1,\ 2x+2y=3$, and solve $x+y+z=1,\ x-y+2z=0$.

The first is inconsistent: the second equation minus twice the first gives $0=1$. The second has rank 2 with 3 unknowns, so it is a line: $\left(\frac{1}{2}-\frac{3}{2}t,\ \frac{1}{2}+\frac{1}{2}t,\ t\right)$.

Watch out

- Using the dot product where a cross product is needed.
- Forgetting that the cross product is anticommutative.
- Using a step that is not an elementary row operation, such as adding a constant to a row.
- Failing to check for inconsistency before back-substitution.

Geometry and linear systems together

Key fact: distances in three dimensions

Three formulas cover most distance problems. The distance from a point P to the plane $ax+by+cz=d$ is $\dfrac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$. The distance from P to the line

through A with direction $\mathbf{d}$ is , because the cross product gives the area of a parallelogram whose base is $|\mathbf{d}|$. The distance between skew lines is : the projection of any connecting vector onto the common perpendicular.

Method: solving and classifying a linear system

1. Write the augmented matrix $(A \mid \mathbf{b})$.
2. Use row operations (swap rows, multiply a row by a non-zero number, add a multiple of one row to another) to reach row echelon form.
3. If a row reads $(0 \ \cdots \ 0 \mid c)$ with $c \neq 0$, the system is inconsistent.
4. Otherwise the number of free parameters is the number of unknowns minus the rank.
5. Solve by back substitution, write the solution in vector form, and check it in every original equation.

Worked example 6: The angle between two planes

Find the acute angle between the planes $2x - y + 2z = 3$ and $x + 2y + 2z = 7$.

The angle between two planes equals the angle between their normals $\mathbf{n}_1 = (2, -1, 2)$ and $\mathbf{n}_2 = (1, 2, 2)$. We have $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 + 4 = 4$ and $|\mathbf{n}_1| = |\mathbf{n}_2| = 3$. So $\cos\theta = \dfrac{4}{9}$ and $\theta \approx 63.6^\circ$. If the dot product had been negative we would take its absolute value to obtain the acute angle.

Worked example 7: A line of intersection

Find a vector equation of the line where $x + y - z = 1$ and $2x - y + z = 5$ meet.

Adding the equations gives $3x = 6$, so $x = 2$ on the whole line. Then $y - z = -1$, so taking $z = t$ we get $y = t - 1$. The line is $\mathbf{r} = (2, -1, 0) + t(0, 1, 1)$. As a check, the direction should be perpendicular to both normals: $(0, 1, 1) \cdot (1, 1, -1) = 0$ and $(0, 1, 1) \cdot (2, -1, 1) = 0$. Equivalently, the direction is a multiple of $(1, 1, -1) \times (2, -1, 1) = (0, -3, -3)$.

Worked example 8: A system in context

A garden centre sells three types of compost bag with masses 10 kg, 20 kg and 40 kg. A delivery of 30 bags weighs 700 kg and contains twice as many 10 kg bags as 40 kg bags. How many of each are there?

Let x, y, z be the numbers of 10, 20 and 40 kg bags. Then $x + y + z = 30$,

$10x + 20y + 40z = 700$ and $x = 2z$. Substituting the last equation: $3z + y = 30$ and $20z + 20y + 40z = 700$, i.e. $60z + 20y = 700$, so $3z + y = 35$. These two equations contradict each other, so no such delivery exists. Changing the total mass to 600 kg gives $3z + y = 30$ twice: infinitely many solutions in theory, but only whole numbers with $y = 30 - 3z \geq 0$ make sense, such as $z = 5$, $x = 10$, $y = 15$.

Worked example 9: Projection and the component perpendicular

Split $\mathbf{v} = (3, 4, 5)$ into a part parallel to $\mathbf{u} = (1, 0, 1)$ and a part perpendicular to it.

The parallel part is $\dfrac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} = \dfrac{8}{2}(1, 0, 1) = (4, 0, 4)$. The perpendicular part is $\mathbf{v} - (4, 0, 4) = (-1, 4, 1)$, and indeed $(-1, 4, 1) \cdot (1, 0, 1) = 0$. This decomposition is used in mechanics to split a force into components along and across a slope.

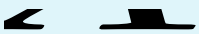
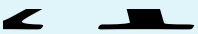
Worked example 10: Rank and the number of solutions

For which values of k does $x + y + kz = 1$, $x + ky + z = 1$, $kx + y + z = 1$ have a unique solution?

The determinant of the coefficient matrix is $-(k - 1)^2(k + 2)$. For $k \neq 1$ and $k \neq -2$ it is non-zero and the solution is unique: by symmetry $x = y = z = \dfrac{1}{k + 2}$. For $k = 1$ all three equations are $x + y + z = 1$: rank 1, two free parameters, a whole plane of solutions. For $k = -2$, adding the three equations gives $0 = 3$: no solution.

Worked example 11: Area of a triangle in space

Find the area of the triangle with vertices $A(1, 1, 0)$, $B(3, 2, 1)$ and $C(2, 4, 2)$, and the angle at A.

The sides from A are  and . Their cross product is $(1 \cdot 2 - 1 \cdot 3,\ 1 \cdot 1 - 2 \cdot 2,\ 2 \cdot 3 - 1 \cdot 1) = (-1, -3, 5)$, with length $\sqrt{1 + 9 + 25} = \sqrt{35}$. The area of the triangle is half the area of the parallelogram: $\frac{1}{2}\sqrt{35} \approx 2.96$. For the angle,  and the lengths are $\sqrt{6}$ and $\sqrt{14}$, so $\cos A = \dfrac{7}{\sqrt{84}} \approx 0.764$ and $A \approx 40.2^\circ$. As a check, $\sin A = \dfrac{\sqrt{35}}{\sqrt{84}} \approx 0.645$ and $0.764^2 + 0.645^2 \approx 1$.

Note: linear systems in the real world

Large linear systems are among the most common calculations performed by computers. Engineers analyse structures with thousands of unknown forces,

economists study the flows between sectors of an economy, and computer graphics uses vectors, dot products and cross products for every rendered frame: the cross product gives the normal to a surface, and the dot product with the light direction decides how bright it looks. Gaussian elimination, the method of this chapter, is still at the heart of the software that solves such problems. For very large systems, computers use refined versions that choose the largest available pivot at each step to limit rounding errors, and iterative methods that improve an approximate solution step by step. Understanding small systems by hand is the best preparation for trusting, and checking, the answers that such software produces.

Watch out: typical mistakes

- Getting the middle component of the cross product wrong: it is $a_3b_1 - a_1b_3$.
- Forgetting that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- Applying a row operation to the coefficients but not to the right-hand side.
- Concluding "no solution" from a zero row $(0\,0\,0 \mid 0)$, which actually signals a free parameter.

Did you know?

The cross product and the dot product were introduced in the 1880s by Josiah Willard Gibbs and Oliver Heaviside to simplify the equations of electromagnetism. Gaussian elimination is named after Carl Friedrich Gauss, who used it around 1810 to compute the orbit of the asteroid Pallas, but the same method appears in the Chinese text The Nine Chapters on the Mathematical Art, written about two thousand years ago.

Otto's tip

After solving, substitute the answer back into every original equation.