Worksheet - Vectors and systems of linear equations

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Dot product, length and angle *(Section A · Skills practice)*

Let $\mathbf{a} = (2, -1, 3)$, $\mathbf{b} = (1, 4, 2)$ and $\mathbf{c} = (1, 2, 0)$.

a. Find $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$. [3 marks]
b. Find the angle between $\mathbf{a}$ and $\mathbf{b}$ to the nearest 0.1°. [2 marks]
c. Show that $\mathbf{a}$ and $\mathbf{c}$ are perpendicular. [1 mark]
d. Find the orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$. [2 marks]

Question 2

Question 2 – The cross product *(Section A · Skills practice)*

Let $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (1, 4, 2)$.

a. Calculate $\mathbf{a} \times \mathbf{b}$. [3 marks]
b. Check that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2 marks]
c. Find the area of the parallelogram and of the triangle with sides $\mathbf{a}$ and $\mathbf{b}$. [3 marks]

Question 3

Question 3 – Lines and planes *(Section A · Skills practice)*

Points $A(1, 0, 2)$, $B(2, 1, 0)$ and $C(0, 1, 1)$ are given.

a. Find a normal vector to the plane ABC and its Cartesian equation. [4 marks]
b. Find where the line $\mathbf{r} = (1, 1, 1) + t(1, -1, 2)$ meets the plane. [3 marks]
c. Find the distance from $P(3, 2, 1)$ to the plane. [2 marks]

Question 4

Question 4 – Gaussian elimination *(Section A · Skills practice)*

Solve the system by reducing the augmented matrix to row echelon form.

$$x + 2y - z = 3, \quad 2x + y + z = 6, \quad 3x - y + 2z = 7$$

a. Write the augmented matrix. [1 mark]
b. Reduce it to echelon form, naming each row operation. [4 marks]

c. Solve by back substitution and check. [3 marks]

Question 5

Question 5 – How many solutions? *(Section A · Skills practice)*

Consider $x + y + z = 1$, $x + 2y + 3z = 2$, $x + 3y + kz = 3$.

a. Reduce the system and show that the last row is $(0, 0, k - 5 \mid 0)$. [3 marks]
b. For which k is there a unique solution? Find it. [2 marks]
c. Describe the solutions when $k = 5$. [2 marks]
d. If the last right-hand side is changed to 4, what happens when $k = 5$? [2 marks]

Question 6

Question 6 – Rank and homogeneous systems *(Section A · Skills practice)*

Let $A =$  .

a. Find the rank of A. [2 marks]
b. Solve $A\mathbf{x} = \mathbf{0}$. [3 marks]
c. Explain how the rank and the number of unknowns give the number of free parameters. [2 marks]

Question 7

Question 7 – Pushing a trolley *(Section B · Real-life practice)*

A shopper pushes a supermarket trolley with a constant force $\mathbf{F} = (3, 4, 0)$ N while it moves from the origin to the point $(5, 2, 0)$ (in metres).

a. Find the displacement vector and its length.
b. The work done is $W = \mathbf{F} \cdot \mathbf{d}$. Find it.
c. Find the angle between the force and the displacement.
d. Find the component of the force in the direction of motion.

Question 8

Question 8 – Mixing a fruit punch *(Section B · Real-life practice)*

A café makes 10 litres of fruit punch from orange (x l), apple (y l) and mango (z l) juice. The costs and sugar contents per litre are given. The punch must cost £12 and contain 1 040 g of sugar.

Juice	Orange	Apple	Mango
Cost per litre (£)	1.20	0.80	2.00

Juice	Orange	Apple	Mango
Sugar per litre (g)	90	100	140

a. Write the three equations.
b. Solve the system by elimination.
c. Check your answer.
d. If the café wants only 1 000 g of sugar, keeping the volume and cost, what changes?

Question 9

Question 9 – Balancing a chemical equation *(Section B · Real-life practice)*

Propane burns in oxygen: $a\,\mathrm{C_3H_8} + b\,\mathrm{O_2} \to c\,\mathrm{CO_2} + d\,\mathrm{H_2O}$.

Element	Left	Right
Carbon	$3a$	c
Hydrogen	$8a$	$2d$
Oxygen	$2b$	$2c + d$

a. Write a homogeneous linear system in a, b, c, d.
b. Explain why the system has infinitely many solutions.
c. Find the smallest positive whole-number solution.
d. How many molecules of oxygen are needed to burn 10 molecules of propane?

Question 10

Question 10 – Traffic around a one-way system *(Section B · Real-life practice)*

Four junctions A, B, C, D lie on a one-way loop with flows x_1 ($A \to B$), x_2 ($B \to C$), x_3 ($C \to D$) and x_4 ($D \to A$), in cars per hour. At each junction the flow in equals the flow out.

Junction	Enters from outside	Leaves to outside
A	400	250
B	200	300
C	350	400
D	300	300

a. Write the four equations.
b. Show that one equation is redundant and solve the system with $x_4 = t$.
c. Roadworks close the section $D \to A$. Find the other flows.
d. What is the smallest possible flow on $A \to B$?

Question 11

Question 11 – The flight of a football *(Section B · Real-life practice)*

A football is kicked from a height of 1 m. Its height h (m) at time t (s) is modelled by $h = a + bt + ct^2$. Measurements give $h(0) = 1$, $h(1) = 6$, $h(2) = 9$.

a. Write a linear system for a, b, c.
b. Solve it.
c. Find the maximum height and when it occurs.
d. When does the ball land?

Question 12

Question 12 – A drone near a power line *(Section B · Real-life practice)*

A straight power line passes through $A(0, 0, 10)$ with direction $\mathbf{d} = (1, 1, 0)$ (metres). A drone hovers at $P(5, 1, 12)$. Safety rules require at least 3 m between drones and power lines.

a. Find and .

b. Use to find the distance from the drone to the line.

c. Find the point of the line closest to the drone.
d. Is the drone breaking the rule?

Question 13

Question 13 – Theatre ticket prices *(Section B · Real-life practice)*

A theatre sells adult, child and senior tickets. Sales and takings for three performances are shown.

Performance	Adults	Children	Seniors	Takings (£)
Friday	100	50	30	2 200
Saturday	80	70	20	1 960
Sunday	120	30	50	2 540

a. Write the system for the prices a, c, s.
b. Solve it by Gaussian elimination.
c. Check with Sunday.
d. The theatre wants £2 500 on Monday with 100 adults and 40 seniors. How many children's tickets must it sell?

Question 14

Question 14 – The ridge of a roof *(Section B · Real-life practice)*

Two faces of a roof lie in the planes $\Pi_1 : x + z = 6$ and $\Pi_2 : y + z = 6$ (in metres, with z the height).

a. Write down normal vectors to the two planes.
b. Find the angle between the planes.
c. Find the direction of the ridge, where the faces meet.
d. Find a vector equation of the ridge line and the height of the ridge above the point $(2, 2, 0)$.

Question 15

Question 15 – Three planes *(Section C · Problem solving)*

An architect's design uses the planes $\Pi_1 : x + y + z = 6$, $\Pi_2 : x - y + 2z = 5$ and $\Pi_3 : 2x + az = b$.

Case	Condition	Geometry
unique point	?	three planes meet at a point
line	?	a sheaf through one line
none	?	a triangular prism

a. Show that $\Pi_1 + \Pi_2$ gives $2x + 3z = 11$.
b. Complete the table of conditions on a and b.
c. Solve the system when $a = 1$, $b = 8$.
d. When $a = 3$, $b = 11$, find a vector equation of the common line.
e. Explain the prism case geometrically.

Question 16

Question 16 – An electric circuit *(Section C · Problem solving)*

In a circuit, a 10 V battery drives currents I_1, I_2, I_3 (in amps) through resistors of 2, 4 and 6 ohms. Kirchhoff's laws give:

Law	Equation
Junction	$I_1 = I_2 + I_3$
Loop 1	$2I_1 + 4I_2 = 10$
Loop 2	$4I_2 - 6I_3 = 0$

a. Write the system in matrix form $A\mathbf{I} = \mathbf{b}$.
b. Solve it.
c. Check the junction law.
d. Find the power $P = 10I_1$ supplied by the battery.

Question 17

Question 17 – The least-squares line *(Section C · Problem solving)*

A sensor gives the readings below. No straight line passes through all four points, so we find the line $y = a + bx$ that minimises the sum of squared errors, by solving the normal equations $A^T A\mathbf{u} = A^T\mathbf{y}$.

x	0	1	2	3
y	1	3	4	4

a. Write the overdetermined system $A\mathbf{u} = \mathbf{y}$ with $\mathbf{u} = (a, b)$.
b. Compute $A^T A$ and $A^T\mathbf{y}$.
c. Solve the normal equations.
d. Find the residuals and their sum of squares.
e. Explain why the residual vector is perpendicular to the columns of A.

Question 18

Question 18 – Volume of a crystal cell *(Section C · Problem solving)*

A crystal is built from cells shaped like parallelepipeds spanned by $\mathbf{u} = (2, 0, 0)$, $\mathbf{v} = (1, 3, 0)$ and $\mathbf{w} = (1, 1, 4)$ (in ångströms).

a. Compute $\mathbf{v} \times \mathbf{w}$ and the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.
b. Check that it equals the determinant of the matrix with rows $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
c. Find the volume of one cell and of the tetrahedron with edges $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
d. Show that $(1, 1, 0)$, $(2, 4, 0)$ and $(3, 5, 0)$ are coplanar, and decide whether $\mathbf{u}$, $\mathbf{v}$ and $(3, 3, 0)$ can span a cell.

Question 19

Question 19 – Two flight paths *(Section C · Problem solving)*

Two aircraft fly along straight paths: $L_1 : \mathbf{r} = s(1, 0, 0)$ and $L_2 : \mathbf{r} = (0, 1, 2) + t(0, 1, 1)$ (in km).

a. Show that the paths are not parallel and do not intersect (they are skew).
b. Find a vector $\mathbf{n}$ perpendicular to both directions.
c. Find the shortest distance between the paths.
d. Find the points where this distance is achieved.
e. Air traffic rules require 1 km of separation. Comment.

Question 20

Question 20 – Challenge: the Cauchy-Schwarz inequality *(Section D · Challenge)*

For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, consider $q(t) = |\mathbf{a} + t\mathbf{b}|^2 \geq 0$ for all real t.

a. Expand $q(t)$ as a quadratic in t.
b. Use the discriminant to prove $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$.
c. When does equality hold?
d. Deduce the triangle inequality $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.
e. Verify both for $\mathbf{a} = (1, 2, 2)$, $\mathbf{b} = (3, 0, 4)$.