Answers - Vectors and systems of linear equations

Year: University Year 1

Answer to Question 1

a. Multiply corresponding components and add:
$\mathbf{a} \cdot \mathbf{b} = 2(1) + (-1)(4) + 3(2) = 2 - 4 + 6 = \mathbf{4}$. The lengths are $|\mathbf{a}| = \sqrt{4 + 1 + 9} = \sqrt{14}$ and $|\mathbf{b}| = \sqrt{1 + 16 + 4} = \sqrt{21}$.

b. From $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$: $\cos\theta = \dfrac{4}{\sqrt{14}\sqrt{21}} = \dfrac{4}{\sqrt{294}} \approx 0.2333$, so $\theta \approx$ **76.5°**. The dot product is positive, so the angle is acute, as expected.

c. $\mathbf{a} \cdot \mathbf{c} = 2(1) + (-1)(2) + 3(0) = 2 - 2 + 0 = 0$. A zero dot product between non-zero vectors means they are perpendicular.

d. The projection is $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \dfrac{4}{21}(1, 4, 2) = \left(\frac{4}{21}, \frac{16}{21}, \frac{8}{21}\right)$: the "shadow" of $\mathbf{a}$ on the line of $\mathbf{b}$.

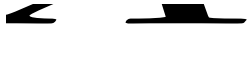
Answer to Question 2

a. Use $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$:
$\mathbf{a} \times \mathbf{b} = \big((-1)(2) - (3)(4),\ (3)(1) - (2)(2),\ (2)(4) - (-1)(1)\big) = (-14, -1, 9)$.

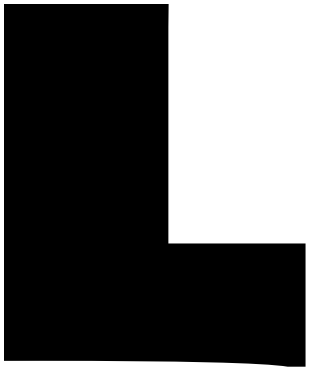
b. $(-14, -1, 9) \cdot (2, -1, 3) = -28 + 1 + 27 = 0$ and
$(-14, -1, 9) \cdot (1, 4, 2) = -14 - 4 + 18 = 0$. Both dot products are zero, so the cross product is perpendicular to $\mathbf{a}$ and to $\mathbf{b}$, as it should be.

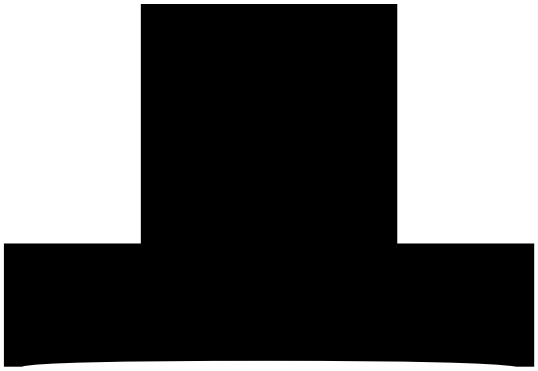
c. The length of the cross product is the area of the parallelogram:
$|\mathbf{a} \times \mathbf{b}| = \sqrt{196 + 1 + 81} = \sqrt{278} \approx$ **16.67** square units. The triangle is half of the parallelogram: $\frac{1}{2}\sqrt{278} \approx$ **8.34** square units.

Answer to Question 3

a. Two direction vectors in the plane are  and .

Their cross product is perpendicular to both:





. The plane is $x + 3y + 2z = d$; substituting A gives $d = 1 + 0 + 4 = 5$, so $x + 3y + 2z = 5$. Check with B: $2 + 3 + 0 = 5$.

b. Substitute the line into the plane: $(1 + t) + 3(1 - t) + 2(1 + 2t) = 6 + 2t = 5$, so

$t = -\frac{1}{2}$. The point is $(0.5,\ 1.5,\ 0)$.

c. Use $d = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|3 + 6 + 2 - 5|}{\sqrt{1 + 9 + 4}} = \dfrac{6}{\sqrt{14}} \approx$ **1.60** units.

Answer to Question 4

a. Each row holds the coefficients of one equation, with the right-hand side after the bar: [redacted].

b. Use row 1 to clear the first column: $R_2 \to R_2 - 2R_1$ gives $(0, -3, 3 \mid 0)$, and $R_3 \to R_3 - 3R_1$ gives $(0, -7, 5 \mid -2)$. Then clear the second column below the pivot: $R_3 \to R_3 - \frac{7}{3}R_2$ gives $(0, 0, -2 \mid -2)$. The matrix is now in echelon form.

c. Work upwards. The last row says $-2z = -2$, so $z = 1$. The second row says $-3y + 3z = 0$, so $y = 1$. The first row gives $x + 2 - 1 = 3$, so $x = 2$. The solution is $(x, y, z) = (2, 1, 1)$. Check: $2 + 2 - 1 = 3$, $4 + 1 + 1 = 6$ and $6 - 1 + 2 = 7$.

Answer to Question 5

a. $R_2 \to R_2 - R_1$ gives $(0, 1, 2 \mid 1)$ and $R_3 \to R_3 - R_1$ gives $(0, 2, k - 1 \mid 2)$. Then $R_3 \to R_3 - 2R_2$ gives $(0, 0, k - 5 \mid 0)$, as required.

b. If $k \neq 5$, the last row gives $(k - 5)z = 0$, so $z = 0$. Then $y + 2z = 1$ gives $y = 1$, and $x + y + z = 1$ gives $x = 0$. The unique solution is $(0, 1, 0)$.

c. If $k = 5$, the last row is $0 = 0$, so $z = t$ is a free parameter. Then $y = 1 - 2t$ and $x = 1 - y - z = t$. There are **infinitely many** solutions $(t,\ 1 - 2t,\ t)$: the three planes meet in a line.

d. With right-hand side 4, the last row becomes $(0, 0, 0 \mid 1)$, which says $0 = 1$. The system is inconsistent: **no solution**.

Answer to Question 6

a. $R_2 \to R_2 - 2R_1$ gives a zero row, because row 2 is twice row 1. $R_3 \to R_3 - R_1$ gives $(0, -2, -2)$. Two non-zero rows remain in echelon form, so **rank 2**.

b. The reduced system is $x + 2y + 3z = 0$ and $-2y - 2z = 0$. From the second, $y = -z$; then $x = -2y - 3z = 2z - 3z = -z$. Letting $z = -t$, the solutions are $t(1, 1, -1)$, $t \in \mathbb{R}$. Check: $A(1, 1, -1)^T = (0, 0, 0)^T$.

c. Each non-zero row fixes one unknown, so the number of free parameters is the number of unknowns minus the rank: $3 - 2 = 1$. One parameter means the solutions form a line through the origin.

Answer to Question 7

a. The trolley moves from $(0, 0, 0)$ to $(5, 2, 0)$, so $\mathbf{d} = (5, 2, 0)$ and $|\mathbf{d}| = \sqrt{25 + 4} = \sqrt{29} \approx$ **5.39 m**.

b. $W = \mathbf{F} \cdot \mathbf{d} = 3(5) + 4(2) + 0 = 15 + 8 =$ **23 J**. Only the part of the force along the motion does work.

c. $|\mathbf{F}| = \sqrt{9 + 16} = 5$, so $\cos\theta = \dfrac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{F}||\mathbf{d}|} = \dfrac{23}{5\sqrt{29}} \approx 0.854$, and $\theta \approx$ **31.3°**.

d. The component of $\mathbf{F}$ along $\mathbf{d}$ is $\dfrac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{d}|} = \dfrac{23}{\sqrt{29}} \approx$ **4.27 N**, out of a total force of 5 N. Equivalently $5\cos 31.3^\circ \approx 4.27$.

Answer to Question 8

a. $x + y + z = 10$, $1.2x + 0.8y + 2z = 12$, $90x + 100y + 140z = 1040$.

b. Multiply the second by 5: $6x + 4y + 10z = 60$; subtract $4\times$ the first: $2x + 6z = 20$, so $x = 10 - 3z$. Divide the third by 10 and subtract $10\times$ the first: $-x + 4z = 4$. Then $-(10 - 3z) + 4z = 4$ gives $7z = 14$, so $z = 2$, $x = 4$, $y = 4$. **4 l orange, 4 l apple, 2 l mango**.

c. $4 + 4 + 2 = 10$, $4.8 + 3.2 + 4 = 12$, $360 + 400 + 280 = 1040$.

d. Now $-x + 4z = 0$; with $x = 10 - 3z$: $7z = 10$, $z \approx 1.43$, $x \approx 5.71$, $y \approx 2.86$ litres.

Answer to Question 9

a. Each element must balance, so left minus right is zero: carbon $3a - c = 0$, hydrogen $8a - 2d = 0$, oxygen $2b - 2c - d = 0$.

b. There are 3 equations in 4 unknowns, so the rank is at most 3 and there is at least one free parameter. This makes sense chemically: if (a, b, c, d) balances the equation, so does any multiple.

c. Take $a = 1$. Then $c = 3$ and $d = 4$, and the oxygen equation gives $2b = 6 + 4$, so $b = 5$. The balanced equation is $\mathrm{C_3H_8} + 5\mathrm{O_2} \to 3\mathrm{CO_2} + 4\mathrm{H_2O}$. Check oxygen: $10 = 6 + 4$.

d. Multiply the solution by 10: $a = 10$ gives $b = 50$, so **50 molecules** of oxygen are needed.

Answer to Question 10

a. A: $x_4 + 400 = x_1 + 250$; B: $x_1 + 200 = x_2 + 300$; C: $x_2 + 350 = x_3 + 400$; D: $x_3 + 300 = x_4 + 300$.

b. Adding the first three gives the fourth, so the rank is 3 and there is one free parameter: $x_1 = 150 + t$, $x_2 = 50 + t$, $x_3 = t$, $x_4 = t$.

c. $t = 0$: $x_1 = 150$, $x_2 = 50$, $x_3 = 0$.

d. Flows cannot be negative, so $t \geq 0$ and $x_1 \geq$ **150 cars per hour**.

Answer to Question 11

a. Substitute each measurement into $h = a + bt + ct^2$: $t = 0$ gives $a = 1$; $t = 1$ gives $a + b + c = 6$; $t = 2$ gives $a + 2b + 4c = 9$.
b. Using $a = 1$: $b + c = 5$ and $2b + 4c = 8$, that is $b + 2c = 4$. Subtracting, $c = -1$, and then $b = 6$. The model is $h = 1 + 6t - t^2$.
c. Complete the square: $h = 1 - (t^2 - 6t) = 10 - (t - 3)^2$. The square is never negative, so the maximum height is **10 m at** $t = 3$ **s**.
d. The ball lands when $h = 0$: $t^2 - 6t - 1 = 0$, so $t = 3 \pm \sqrt{10}$. Only the positive root makes sense: $t = 3 + \sqrt{10} \approx$ **6.16 s**.

Answer to Question 12

a.  . Then .

b. The cross product has length , so dividing by $|\mathbf{d}|$ gives the perpendicular distance: $\dfrac{\sqrt{4+4+16}}{\sqrt{2}} = \sqrt{12} \approx$ **3.46 m**.

c. The closest point is $A + t\mathbf{d}$ with , which is $(3, 3, 10)$.

Check: the distance from P is $\sqrt{4+4+4} = \sqrt{12}$, as in part (b).

d. Since $3.46 > 3$, the drone is **not** breaking the rule, but the margin is less than half a metre.

Answer to Question 13

a. Each performance gives one equation: $100a + 50c + 30s = 2200$, $80a + 70c + 20s = 1960$, $120a + 30c + 50s = 2540$. Dividing each by 10 gives smaller numbers: $10a + 5c + 3s = 220$, $8a + 7c + 2s = 196$, $12a + 3c + 5s = 254$.
b. Eliminating a from the second and third equations using the first, and then eliminating c, leaves one equation in s, giving $s = 10$. Back substitution gives $c = 8$ and $a = 15$. The prices are $a = £15$, $c = £8$, $s = £10$. Check Friday: $1500 + 400 + 300 = 2200$.
c. Sunday: $120(15) + 30(8) + 50(10) = 1800 + 240 + 500 = 2540$, which matches the

takings.

d. 100 adults and 40 seniors bring in $1500 + 400 = 1900$, leaving $2500 - 1900 = 600$. At £8 each, the theatre must sell $600/8 =$ **75 children's tickets**.

Answer to Question 14

a. The coefficients of x, y, z give the normals: $\mathbf{n}_1 = (1, 0, 1)$ and $\mathbf{n}_2 = (0, 1, 1)$.

b. The angle between the planes is the angle between their normals: $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \dfrac{1}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$, so the angle is **60°**.

c. The ridge lies in both planes, so it is perpendicular to both normals: $\mathbf{n}_1 \times \mathbf{n}_2 = (0 \cdot 1 - 1 \cdot 1,\ 1 \cdot 0 - 1 \cdot 1,\ 1 \cdot 1 - 0 \cdot 0) = (-1, -1, 1)$.

d. The point $(6, 6, 0)$ satisfies both equations, so the ridge is $\mathbf{r} = (6, 6, 0) + t(-1, -1, 1)$. Above $(2, 2, 0)$ we need $6 - t = 2$, so $t = 4$, and the height of the ridge there is **4 m**. Check: $2 + 4 = 6$ in both planes.

Answer to Question 15

a. Adding the equations: $2x + 0y + 3z = 11$.

b. Unique point: $a \neq 3$. Line: $a = 3$**,** $b = 11$. None: $a = 3$**,** $b \neq 11$.

c. $2x + 3z = 11$ and $2x + z = 8$: $z = 1.5$, $x = 3.25$, $y = 6 - 4.75 = 1.25$: $(3.25,\ 1.25,\ 1.5)$.

d. Let $z = t$: $x = \dfrac{11 - 3t}{2}$, $y = 6 - x - t = \dfrac{1 + t}{2}$: $\mathbf{r} = \left(\frac{11}{2}, \frac{1}{2}, 0\right) + t\left(-\frac{3}{2}, \frac{1}{2}, 1\right)$.

e. Each pair of planes meets in a line, and the three lines are parallel but do not coincide.

Answer to Question 16

a. Rewrite the junction law as $I_1 - I_2 - I_3 = 0$. Then $A =$  and $\mathbf{b} = (0, 10, 0)$.

b. Loop 2 gives $I_3 = \frac{2}{3}I_2$. The junction law then gives $I_1 = I_2 + \frac{2}{3}I_2 = \frac{5}{3}I_2$. Substituting into loop 1: $\frac{10}{3}I_2 + 4I_2 = 10$, so $\frac{22}{3}I_2 = 10$ and $I_2 = \dfrac{15}{11}$. Hence $I_1 = \dfrac{25}{11} \approx 2.27$**,** $I_2 \approx 1.36$**,** $I_3 = \dfrac{10}{11} \approx 0.91$ **A**.

c. $I_2 + I_3 = \dfrac{15}{11} + \dfrac{10}{11} = \dfrac{25}{11} = I_1$, so the junction law holds.

d. $P = 10I_1 = \dfrac{250}{11} \approx$ **22.7 W** supplied by the battery.

Answer to Question 17

a. Rows $(1, x_i)$: $A =$ , $\mathbf{y} = (1, 3, 4, 4)$.

b. $A^T A = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$, $A^T \mathbf{y} = (12, 23)$.

c. Determinant $56 - 36 = 20$: $a = \dfrac{14 \times 12 - 6 \times 23}{20} = 1.5$, $b = \dfrac{4 \times 23 - 6 \times 12}{20} = 1$: $y = 1.5 + x$.

d. Residuals $-0.5,\ 0.5,\ 0.5,\ -0.5$; sum of squares **1**.

e. The normal equations say $A^T(\mathbf{y} - A\mathbf{u}) = \mathbf{0}$: the error is orthogonal to the column space, so $A\mathbf{u}$ is the projection of $\mathbf{y}$.

Answer to Question 18

a. $\mathbf{v} \times \mathbf{w} = (12, -4, -2)$; $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) =$ **24**.

b. $\det = 2(12 - 0) - 0 + 0 = 24$.

c. Cell: **24 Å**^{3}; tetrahedron: $\frac{1}{6} \times 24 =$ **4 Å**^{3}.

d. All three have $z = 0$, so the triple product is 0. Likewise $\mathbf{u}$, $\mathbf{v}$, $(3, 3, 0)$ lie in the plane $z = 0$: the volume is 0, so **no**.

Answer to Question 19

a. The directions are not multiples. Equating: $s = 0$, $0 = 1 + t$, $0 = 2 + t$: impossible, so they do not meet.

b. $\mathbf{n} = (1, 0, 0) \times (0, 1, 1) = (0, -1, 1)$.

c. With $A = (0, 0, 0)$, $B = (0, 1, 2)$:  $\dfrac{1}{\sqrt{2}} \approx 0.707$ **km**.

d. $P_2 - P_1 = (-s, 1 + t, 2 + t)$ must be perpendicular to both directions: $s = 0$ and $2t + 3 = 0$, $t = -1.5$. Points $(0, 0, 0)$ **and** $(0, -0.5, 0.5)$.

e. 0.707: the separation would be broken if both aircraft were at those points at the same time; but the times also matter.

Answer to Question 20

a. $q(t) = |\mathbf{b}|^2 t^2 + 2(\mathbf{a} \cdot \mathbf{b})t + |\mathbf{a}|^2$.

b. A quadratic that is never negative has discriminant ≤ 0: $4(\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2 \leq 0$.

c. When the discriminant is 0: q has a root, so $\mathbf{a} = -t\mathbf{b}$, i.e. the vectors are parallel.

d. $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 \leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)^2$.
e. $\mathbf{a}\cdot\mathbf{b} = 11 \leq 3 \times 5 = 15$; $|\mathbf{a}+\mathbf{b}| = |(4,2,6)| = \sqrt{56} \approx 7.48 \leq 8$.