Test answers - Vector spaces and linear maps

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

U: contains $\mathbf{0}$ (1); if $x = 2y$ and $x' = 2y'$ then $x + x' = 2(y + y')$ and $\lambda x = 2\lambda y$ (1): subspace (1). V: $(1, 0) \in V$ but $-1 \cdot (1, 0) = (-1, 0) \notin V$ (1): not closed under scalar multiplication, so not a subspace.

For U, the three subspace conditions are checked: the zero vector satisfies $0 = 2 \times 0$, and the set is closed under addition and under scalar multiplication because the condition $x = 2y$ is linear. Geometrically, U is a plane through the origin in $\mathbb{R}^3$. For V, one counterexample is enough to show it is not a subspace; V is a closed half-plane, closed under addition but not under multiplication by negative scalars.

Answer to Question 2 (4 marks)

The determinant of the matrix with these rows is $1(18 - 12) - 1(9 - 3) + 1(4 - 2) = 6 - 6 + 2 = 2 \neq 0$ (2): a basis. Solve $a + b + c = 3$, $a + 2b + 4c = 7$, $a + 3b + 9c = 13$ (one equation for each coordinate): subtract: $b + 3c = 4$, $b + 5c = 6$ so $c = 1$, $b = 1$, $a = 1$ (2). Coordinates $(1, 1, 1)$.

Three vectors in the three-dimensional space $\mathbb{R}^3$ form a basis exactly when they are linearly independent, which is equivalent to a non-zero determinant. In the elimination, subtracting the first equation from the second and third gives $b + 3c = 4$ and $2b + 8c = 10$, that is $b + 4c = 5$; a further subtraction gives $c = 1$. Check: $1 \times (1, 1, 1) + 1 \times (1, 2, 3) + 1 \times (1, 4, 9) = (3, 7, 13)$.

Answer to Question 3 (4 marks)

a. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ (1).

b. Kernel: $x = -y$, $z = -y$, spanned by $(-1, 1, -1)$ (1). Image: the columns $(1, 0)$, $(1, 1)$ span $\mathbb{R}^2$, basis $\{(1, 0), (0, 1)\}$ (1).

c. Rank 2 + nullity 1 $= 3$ (1).

In (a), the columns of the matrix are the images of the standard basis vectors: $T(1, 0, 0) = (1, 0)$, $T(0, 1, 0) = (1, 1)$ and $T(0, 0, 1) = (0, 1)$. In (b), the kernel is the solution set of $x + y = 0$ and $y + z = 0$; taking $y = 1$ as the free variable gives the spanning vector. Check: $T(-1, 1, -1) = (0, 0)$. In (c), the dimension of the domain $\mathbb{R}^3$ is 3, which equals rank plus nullity.

Answer to Question 4 (4 marks)

a. $5 - 2 = 3$ (1).
b. The image has dimension 2 (1), less than 3, so it cannot cover $\mathbb{R}^3$ (1).
c. Not injective since the kernel is non-trivial (nullity 3) (1).

In (a), the rank-nullity theorem says rank plus nullity equals the dimension of the domain, which is 5 because A acts on $\mathbb{R}^5$. In (b), the rank is the dimension of the image, a plane inside $\mathbb{R}^3$, so the map is not surjective. In (c), a linear map is injective exactly when its kernel is $\{\mathbf{0}\}$; here three independent directions are sent to zero, so different inputs can give the same measurement.

Answer to Question 5 (4 marks)

a. Coordinates $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ (1); determinant $2 \neq 0$ (1).
b. $a + c = 2$, $a + b = 3$, $b + c = 1$ (1): sum $a + b + c = 3$, so $b = 1$, $c = 0$, $a = 2$.
$q = 2p_1 + p_2$ (1). Check: $2 + 2x + x + x^2 = 2 + 3x + x^2$.

In (a), the coordinate vector lists the coefficients of 1, x and x^2: for example $p_2 = 0 + 1x + 1x^2$. Three vectors in the three-dimensional space P_2 form a basis when the determinant of their coordinates is non-zero; here it equals $1(1 - 0) - 0 + 1(1 - 0) = 2$. In (b), compare the coefficients of 1, x and x^2 in $ap_1 + bp_2 + cp_3 = q$. Adding the three equations gives $2(a + b + c) = 6$.