Lesson - Vector spaces and linear maps

Year: University Year 1

Vector spaces abstract the properties of vectors so that one set of theorems applies to arrows, polynomials, functions and matrices alike.

Starter activity

Take a flat piece of card and imagine it extends forever through the origin of three-dimensional space. Which points can you reach by adding multiples of two arrows lying in the card? What if you add a third arrow that is not in the card? What if the third arrow lies in the card? Sketch each situation and count how many arrows you really need to describe every point.

Vector spaces and subspaces

Definition: vector space

A **vector space** over $\mathbb{R}$ is a set V with addition and scalar multiplication satisfying eight axioms: addition is commutative and associative, there is a zero vector, every vector has an additive inverse, and scalar multiplication is compatible and distributes over both additions, with $1\mathbf{v} = \mathbf{v}$. Examples: $\mathbb{R}^n$; the polynomials P_n of degree at most n; continuous functions on $[0,1]$; $m \times n$ matrices.

Definition: subspace

A subset $W \subseteq V$ is a **subspace** if (i) $\mathbf{0} \in W$, (ii) $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u}+\mathbf{v} \in W$, (iii) $\mathbf{v} \in W,\ c \in \mathbb{R} \Rightarrow c\mathbf{v} \in W$.

Example 1

Which of $\{x+y+z=0\}$, $\{xy=0\}$ and $\{x \geq 0\}$ are subspaces?

The plane $\{(x,y,z) : x+y+z=0\}$ is a subspace of $\mathbb{R}^3$: if two vectors have coordinates summing to 0, so do their sum and any multiple. $\{(x,y) : xy=0\}$ is not, since $(1,0)+(0,1)=(1,1)$ is not in it. $\{(x,y) : x \geq 0\}$ fails (iii) with $c=-1$.

Span, independence and basis

Definition: span, independence, basis, dimension

The **span** of $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the set of all linear combinations; it is a subspace. The vectors are **linearly independent** if $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ forces every $c_i = 0$. A **basis** is a linearly independent spanning set. Every basis of V has the same number

of elements, the **dimension** $\dim V$. So $\dim \mathbb{R}^n = n$ and $\dim P_3 = 4$ (basis $1, x, x^2, x^3$).

Method: testing for a basis of $\mathbb{R}^n$

1. Write the n vectors as the columns of a matrix.
2. Compute the determinant.
3. If $\det \neq 0$ the vectors are independent and form a basis; if $\det = 0$ they are dependent.
4. To prove independence in general, start with "suppose $\sum c_i \mathbf{v}_i = \mathbf{0}$" and show every $c_i = 0$.

Example 2

Do $(1,2,3)$, $(0,1,4)$, $(5,6,0)$ form a basis of $\mathbb{R}^3$? What about $(1,2,3)$, $(4,5,6)$, $(7,8,9)$?

The first set has determinant 1, so it is a basis. In the second, $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so the vectors are dependent.

Example 3: coordinates

Find the coordinates of $(2,3,4)$ in the basis $(1,1,0), (0,1,1), (1,0,1)$.

Solve $a + c = 2$, $a + b = 3$, $b + c = 4$. The coordinates are $(\frac{1}{2}, \frac{5}{2}, \frac{3}{2})$; they are unique for each vector.

Linear maps

Definition: linear map

A function $T : V \to W$ is **linear** if $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$ and $T(c\mathbf{v}) = cT\mathbf{v}$.

Rule: matrix of a linear map

Every linear map $\mathbb{R}^n \to \mathbb{R}^m$ is multiplication by an $m \times n$ matrix whose columns are the images of the standard basis vectors. Composition corresponds to matrix multiplication.

Example 4

Find the matrix of $T(x,y) = (x + y, 2x - y)$, and of differentiation $D : P_2 \to P_2$.

T has matrix $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$. In the basis $1, x, x^2$, D has matrix  .

Kernel, image and rank-nullity

Definition: kernel and image

The **kernel** $\ker T = \{\mathbf{v} : T\mathbf{v} = \mathbf{0}\}$ and the **image** $\operatorname{im} T = \{T\mathbf{v}\}$ are subspaces of V and W. Their dimensions are the **nullity** and the **rank**.

Theorem: rank-nullity theorem

$\dim V = \operatorname{rank} T + \operatorname{nullity} T$. Also T is injective if and only if $\ker T = \{\mathbf{0}\}$, and surjective if and only if $\operatorname{im} T = W$.

Method: kernel and image of a matrix map

1. To find the kernel solve $A\mathbf{x} = \mathbf{0}$ by row reduction.
2. The image is the column space of A: find its dimension from the pivot columns.
3. Check rank + nullity equals the number of columns.

Example 5

For $T(x, y, z) = (x + y, y + z)$ find the kernel, image, rank and nullity.

The kernel is spanned by $(1, -1, 1)$ (nullity 1) and the image is all of $\mathbb{R}^2$ (rank 2). Check: $3 = 2 + 1$.

Watch out

- Forgetting to check that the zero vector is in a proposed subspace.
- Claiming independence from pairwise tests only.
- Mixing up kernel (inputs mapped to zero) with image (outputs).
- Forgetting that dimension counts basis vectors, not coordinates.

Working with abstract spaces

Key fact: what a basis does for you

A basis is a minimal list that spans a space, and also a maximal list that is independent. Every vector in the space can be written in exactly one way as a combination of basis vectors, and the coefficients are its coordinates. This makes a basis a coordinate system: it lets you replace an abstract object, such as a polynomial, a matrix or a function, by an ordinary column of numbers. All bases of the same space have the same number of vectors, called the dimension. Choosing a good basis can make a problem easy: the map T in the tanks problem looks complicated in the standard basis but is simply "multiply by 4 and by 2" in a basis of diagonals.

Method: finding the kernel and image of a matrix

1. To find the kernel, row-reduce A and solve $A\mathbf{x} = \mathbf{0}$; the free variables give the basis vectors of the kernel.
2. To find the image, row-reduce A and look at the pivot columns; the corresponding columns of the original matrix form a basis of the column space.
3. The rank is the number of pivots, and the nullity is the number of free variables.
4. Check that rank $+$ nullity equals the number of columns.

Worked example 6: Testing for a basis with a determinant

Do $(1, 0, 2)$, $(2, 1, 0)$ and $(0, 3, 1)$ form a basis of $\mathbb{R}^3$?

Three vectors in $\mathbb{R}^3$ are a basis exactly when they are linearly independent, which happens exactly when the determinant of the matrix of their coordinates is non-zero.

Here [redacted], so they do form a basis. If the determinant had been zero we would look for a dependence relation between them.

Worked example 7: The span of two vectors

Describe the span of $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (0, 1, 1)$ in $\mathbb{R}^3$ and decide whether $(2, 5, 7)$ lies in it.

The vectors are not multiples of each other, so their span is a plane through the origin. A vector is in the span if $(2, 5, 7) = \lambda\mathbf{a} + \mu\mathbf{b}$. The first coordinate gives $\lambda = 2$; the second then gives $4 + \mu = 5$, so $\mu = 1$; the third coordinate gives $6 + 1 = 7$, which agrees. So yes: $(2, 5, 7) = 2\mathbf{a} + \mathbf{b}$. For $(2, 5, 8)$ the same method would fail at the third coordinate, so that vector is outside the plane.

Worked example 8: The kernel of a map between polynomial spaces

Let $T : P_2 \to P_2$ be $T(p) = p'' + p$. Find its kernel and image.

Take $p = a + bx + cx^2$. Then $p'' = 2c$, so $T(p) = (a + 2c) + bx + cx^2$. Setting this equal to zero forces $b = 0$, $c = 0$ and then $a = 0$, so the kernel is $\{0\}$. By rank-nullity the rank is 3, so the image is all of P_2: for any target polynomial one can solve for a, b, c. A map with trivial kernel is injective, and here also surjective, so it is an isomorphism.

Worked example 9: Rotation and its matrix

Find the matrix of the rotation through 90° anticlockwise in the plane, and its kernel and image.

The images of the basis vectors are $(1, 0) \mapsto (0, 1)$ and $(0, 1) \mapsto (-1, 0)$. These are the

columns of the matrix, so it is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Its determinant is 1 and no non-zero vector is sent to zero, so the kernel is $\{\mathbf{0}\}$ and the image is all of $\mathbb{R}^2$. The map is bijective, and its fourth power is the identity, because four quarter-turns make a full turn.

Worked example 10: Rank-nullity for a projection

The map $P(x, y, z) = (x, y, 0)$ projects space onto the horizontal plane. Find its rank and nullity.

The image is the xy-plane, which has dimension 2, so the rank is 2. The kernel is the z-axis, since $P(0, 0, z) = \mathbf{0}$, which has dimension 1. Check: $2 + 1 = 3 = \dim \mathbb{R}^3$. Notice that applying P twice gives the same result as applying it once, which is the algebraic signature of a projection.

Note: why abstract spaces matter

Many objects that look unrelated obey the same rules of addition and scalar multiplication: arrows in the plane, columns of numbers, polynomials, matrices, sound waves and solutions of linear differential equations. Proving a theorem once for an abstract vector space proves it for all of them at the same time. This is why rank-nullity applies equally to a system of equations, to a differentiation operator on polynomials, and to the decoding of a signal.

Watch out: common mistakes

- Checking only that a set is closed under addition. A subspace must also contain the zero vector and be closed under scalar multiplication.
- Confusing "n vectors in $\mathbb{R}^n$ span" with "n vectors in $\mathbb{R}^m$ span": the number of vectors must match the dimension of the space.
- Taking the pivot columns of the reduced matrix, instead of the original matrix, as a basis of the image.
- Assuming that a list with more vectors than the dimension can still be independent: in $\mathbb{R}^n$ any list of more than n vectors is dependent.
- Forgetting that a linear map must satisfy $T(\mathbf{0}) = \mathbf{0}$; translations are not linear.

Did you know?

The idea of a vector space was made precise by Giuseppe Peano in 1888, but its power was seen only in the twentieth century, when Stefan Banach and others used infinite-dimensional spaces of functions to study differential equations. Quantum mechanics is written entirely in the language of vector spaces and linear maps: the state of a particle is a vector, and every measurement is a linear map.

Otto's tip

To prove independence always start with "suppose $\sum c_i \mathbf{v}_i = \mathbf{0}$" and show every $c_i = 0$.