Worksheet - Vector spaces and linear maps

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Testing for subspaces *(Section A · Skills practice)*

A subset W of $\mathbb{R}^n$ is a subspace if it contains $\mathbf{0}$, and is closed under addition and scalar multiplication. Decide whether each set is a subspace of $\mathbb{R}^3$ or $\mathbb{R}^2$, with a proof or a counterexample.

a. $W_1 = \{(x, y, z) : x + y + z = 0\}$ [4 marks]
b. $W_2 = \{(x, y) : xy = 0\}$ [3 marks]
c. $W_3 = \{(x, y, z) : x + y + z = 1\}$ [2 marks]

Question 2

Question 2 – Span and independence *(Section A · Skills practice)*

Answer each part.

a. Is $(3, 5)$ in the span of $(1, 1)$ and $(1, 2)$? [3 marks]
b. Are $(1, 2, 3)$, $(0, 1, 4)$ and $(5, 6, 0)$ linearly independent? [3 marks]
c. Show that $(1, 2, 3)$, $(4, 5, 6)$ and $(7, 8, 9)$ are linearly dependent by finding a relation. [3 marks]

Question 3

Question 3 – Bases and coordinates *(Section A · Skills practice)*

Let $\mathbf{b}_1 = (1, 1, 0)$, $\mathbf{b}_2 = (0, 1, 1)$ and $\mathbf{b}_3 = (1, 0, 1)$.

a. Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$. [3 marks]
b. Find the coordinates of $(2, 3, 4)$ in this basis. [4 marks]
c. What is the dimension of $\mathbb{R}^3$? How do you know it from the basis? [1 mark]

Question 4

Question 4 – Dimensions *(Section A · Skills practice)*

State the dimension of each vector space and justify it by giving a basis.

a. P_3, the polynomials of degree at most 3. [2 marks]
b. $M_{2\times 3}$, the 2×3 matrices. [2 marks]

c. $W = \{(x, y, z) : x + y + z = 0\}$. [3 marks]
d. The symmetric 2×2 matrices. [3 marks]

Question 5

Question 5 – Matrices of linear maps *(Section A · Skills practice)*

Answer each part.

a. Show that $T(x, y) = (x + y,\ 2x - y)$ is linear and write its matrix. [4 marks]
b. Find $\det$ of this matrix and say whether T is invertible. [2 marks]
c. Show that $S(x, y) = (x + 1, y)$ is not linear. [2 marks]

Question 6

Question 6 – Kernel, image and rank-nullity *(Section A · Skills practice)*

Let $A =$  and $T(\mathbf{x}) = A\mathbf{x}$.

a. Find a basis for the kernel of T. [4 marks]
b. Find a basis for the image of T. [3 marks]
c. Verify the rank-nullity theorem. [2 marks]

Question 7

Question 7 – Mixing paints *(Section B · Real-life practice)*

A printer mixes three base inks whose colour vectors (in arbitrary units) are $\mathbf{p}_1 = (2, 1, 0)$, $\mathbf{p}_2 = (0, 1, 1)$ and $\mathbf{p}_3 = (1, 0, 1)$. A mix $a\mathbf{p}_1 + b\mathbf{p}_2 + c\mathbf{p}_3$ uses non-negative amounts.

a. Show that the inks are linearly independent.
b. Find the amounts needed to produce the target colour $(5, 4, 3)$.
c. Explain why a unique recipe exists for every colour vector.
d. Which target colours need a negative amount of some ink? Give an example.

Question 8

Question 8 – Planning a diet *(Section B · Real-life practice)*

Three foods give the following amounts of (protein, carbohydrate, fat) per serving: $\mathbf{f}_1 = (1, 2, 0)$, $\mathbf{f}_2 = (0, 1, 2)$ and $\mathbf{f}_3 = (1, 0, 1)$ (in tens of grams).

a. Show that $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is a basis of $\mathbb{R}^3$.
b. Find the servings that give $(3, 7, 7)$.
c. Describe the span of $\mathbf{f}_1$ and $\mathbf{f}_2$ alone: what shape is it?

d. Could the diet $(3, 7, 7)$ be produced from $\mathbf{f}_1$ and $\mathbf{f}_2$ only?

Question 9

Question 9 – Sensor channels *(Section B · Real-life practice)*

A monitoring system has six sensor channels. A 4×6 matrix A transforms the channel readings $\mathbf{x} \in \mathbb{R}^6$ into four summary values $A\mathbf{x} \in \mathbb{R}^4$. An engineer finds that $\operatorname{rank} A = 3$.

a. What is the dimension of the image of A?
b. Use rank-nullity to find the nullity.
c. Interpret the nullity: how many independent directions of change in the readings are invisible in the summary values?
d. Can the summary values $A\mathbf{x}$ cover all of $\mathbb{R}^4$? Explain.

Question 10

Question 10 – Polynomials as vectors: the height of a ball *(Section B · Real-life practice)*

The height of a ball is the polynomial $h(t) = h_0 + v_0 t - 4.9t^2$, which can be seen as a vector $(h_0, v_0, -4.9)$ in P_2 with basis $\{1, t, t^2\}$.

a. Write the matrix of the differentiation map $D : P_2 \to P_2$ with respect to this basis.
b. Find the kernel and image of D and verify rank-nullity.
c. Find $D(h)$ for $h = 2 + 20t - 4.9t^2$ using the matrix.
d. What physical quantity does $D(h)$ represent? What does $D^2 h$ give?

Question 11

Question 11 – Slanting text *(Section B · Real-life practice)*

To make italic text, a graphics program applies the shear $S(x, y) = (x + 0.5y,\ y)$ to each point of a letter.

a. Write the matrix of S.
b. Find the images of the corners $(0, 0)$, $(1, 0)$, $(0, 2)$ and $(1, 2)$ of a rectangle, and describe the image.
c. Find $\det S$ and explain what it says about the area of the letters.
d. Find S^{-1} and describe what it does.

Question 12

Question 12 – A change of basis on the board *(Section B · Real-life practice)*

On a chessboard-style grid a bishop moves along the diagonals. It is natural to use the basis $\mathcal{B} = \{\mathbf{d}_1, \mathbf{d}_2\}$ with $\mathbf{d}_1 = (1, 1)$ and $\mathbf{d}_2 = (1, -1)$.

a. Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$.
b. Find the coordinates of the square $(3, 1)$ in this basis.
c. Find the coordinates of $(2, 6)$.
d. The reflection in the line $y = x$ swaps x and y. Find its matrix in the basis $\mathcal{B}$.

Question 13

Question 13 – The lighthouse beam *(Section B · Real-life practice)*

A lighthouse beam turns anticlockwise by 90° every minute. On a map with the lighthouse at the origin, the beam direction is a vector in $\mathbb{R}^2$ and each minute the map R is applied to it.

a. Find the matrix of R.
b. Find the kernel and the image of R. Is R an isomorphism?
c. The beam points along $(3, 1)$ now. Find its direction after 1, 2, 3 and 4 minutes.
d. Explain why R^4 is the identity map.

Question 14

Question 14 – Balancing a chemical equation *(Section B · Real-life practice)*

Propane burns in oxygen: $a\,\mathrm{C_3H_8} + b\,\mathrm{O_2} \to c\,\mathrm{CO_2} + d\,\mathrm{H_2O}$. Balancing the atoms gives a homogeneous linear system.

a. Show that the conditions are $3a - c = 0$, $8a - 2d = 0$ and $2b - 2c - d = 0$.
b. Write the system as $A\mathbf{x} = \mathbf{0}$ and find the kernel of A.
c. What is the dimension of the kernel? Why does it make sense to expect it?
d. Write the smallest whole-number balanced equation.

Question 15

Question 15 – Intersecting planes *(Section C · Problem solving)*

Two flat walls of a model building are represented by the subspaces $U = \{(x, y, z) : x + y + z = 0\}$ and $W = \{(x, y, z) : x = y\}$ of $\mathbb{R}^3$.

a. Find a basis of U and a basis of W.
b. Find $U \cap W$ and its dimension.
c. Use the formula $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ to find $\dim(U + W)$.
d. Show that $U \cup W$ is not a subspace.

Question 16

Question 16 – Fitting a curve through data *(Section C · Problem solving)*

A shop records its sales (in hundreds) in three consecutive months: 1, 3 and 7. We

model the sales by a polynomial $p(t) = a + bt + ct^2$ with $t = 0, 1, 2$. Consider the map $T : P_2 \to \mathbb{R}^3$, $T(p) = (p(0), p(1), p(2))$.

Month t	0	1	2
Sales	1	3	7

a. Show that T is linear and write its matrix with respect to the basis $\{1, t, t^2\}$.
b. Show that T is an isomorphism by computing the determinant.
c. Find p with $T(p) = (1, 3, 7)$.
d. Use p to predict sales in month 3. Comment on the reliability.

Question 17

Question 17 – A sensor network *(Section C · Problem solving)*

Four sensors send signals $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Three monitors compute $A\mathbf{x}$ with $A =$ [redacted].

a. Show that the third row is the sum of the first two and deduce the rank of A.
b. Find a basis of the kernel.
c. Find a basis of the image.
d. Verify rank-nullity. What does the kernel mean for the monitors?

Question 18

Question 18 – Independent signals *(Section C · Problem solving)*

A signal generator produces the functions $f_1(t) = 1$, $f_2(t) = \cos t$ and $f_3(t) = \sin t$. These are vectors in the space of continuous functions on $\mathbb{R}$.

a. Suppose $a + b\cos t + c\sin t = 0$ for all t. Substitute $t = 0,\ \frac{\pi}{2},\ \pi$ to show that $a = b = c = 0$.
b. What does this prove about f_1, f_2, f_3?
c. Show that $g(t) = \cos t + \sin t$ lies in the span of f_2, f_3 and write its coordinates.
d. The solution set of $y'' + y = 0$ is a subspace of dimension 2. Which pair from the list is a basis?

Question 19

Question 19 – Two connected tanks *(Section C · Problem solving)*

Two tanks exchange water. The amounts of dye (x, y) change in one step by the linear map $T(x, y) = (3x + y,\ x + 3y)$. Use the basis $\mathcal{B} = \{(1, 1), (1, -1)\}$.

a. Find $T(1, 1)$ and $T(1, -1)$ and write each as a multiple of a basis vector.
b. Write the matrix of T in the basis $\mathcal{B}$.

c. Find $T^5(2,0)$ by writing $(2,0)$ in the basis $\mathcal{B}$.
d. Interpret the two multipliers.

Question 20

Question 20 – Challenge: independence of sums *(Section D · Challenge)*

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be linearly independent vectors in a real vector space. Prove that $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{w}+\mathbf{u}$ are also linearly independent.

a. Suppose $\alpha(\mathbf{u}+\mathbf{v})+\beta(\mathbf{v}+\mathbf{w})+\gamma(\mathbf{w}+\mathbf{u})=\mathbf{0}$. Collect terms in $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
b. Use the independence of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to get three equations for α, β, γ.
c. Solve them and finish the proof.
d. Express the change of vectors by a matrix and find its determinant. Why does the proof fail over arithmetic modulo 2?