Answers - Vector spaces and linear maps

Year: University Year 1

Answer to Question 1

a. $\mathbf{0} = (0, 0, 0)$ satisfies the equation. If (x, y, z) and (x', y', z') are in W_1 then the sum has coordinate total $(x + y + z) + (x' + y' + z') = 0$; and $\lambda(x, y, z)$ has total $\lambda \cdot 0 = 0$. So W_1 **is a subspace** (a plane through the origin).
b. $(1, 0)$ and $(0, 1)$ are in W_2 (since $1 \cdot 0 = 0$ and $0 \cdot 1 = 0$), but their sum $(1, 1)$ has $xy = 1 \neq 0$. Closure under addition fails: **not a subspace**.
c. $(0, 0, 0)$ gives total $0 \neq 1$, so the zero vector is missing: **not a subspace** (it is a plane not through the origin).

Answer to Question 2

a. Solve $a(1, 1) + b(1, 2) = (3, 5)$: $a + b = 3$, $a + 2b = 5$. Subtracting, $b = 2$ and $a = 1$. So yes: $(3, 5) = 1(1, 1) + 2(1, 2)$.
b. The determinant of the matrix with these rows is $1(0 - 24) - 2(0 - 20) + 3(0 - 5) = 1 \neq 0$, so they are **independent**.
c. Try $(7, 8, 9) = a(1, 2, 3) + b(4, 5, 6)$: $a + 4b = 7$, $2a + 5b = 8$ give $b = 2$, $a = -1$, and the third coordinate $-3 + 12 = 9$ works. So $(1, 2, 3) - 2(4, 5, 6) + (7, 8, 9) = \mathbf{0}$: **dependent**.

Answer to Question 3

a. The determinant of the matrix with rows $\mathbf{b}_i$ is $1(1 \cdot 1 - 0) - 1(0 \cdot 1 - 0) + 0 = \ldots$ computing: [redacted]. Three independent vectors in $\mathbb{R}^3$ form a basis.
b. Solve $a\mathbf{b}_1 + b\mathbf{b}_2 + c\mathbf{b}_3 = (2, 3, 4)$: $a + c = 2$, $a + b = 3$, $b + c = 4$. Adding: $2(a + b + c) = 9$, so $a + b + c = 4.5$. Then $b = 4.5 - 2 = 2.5$, $c = 4.5 - 3 = 1.5$, $a = 4.5 - 4 = 0.5$. Coordinates: $(0.5,\ 2.5,\ 1.5)$.
c. The dimension is the number of vectors in a basis: **3**.

Answer to Question 4

a. Basis $1, x, x^2, x^3$: dimension **4**.
b. One basis matrix for each of the 6 positions (a 1 in that position, 0 elsewhere): dimension **6**.
c. $(x, y, z) = (-y - z, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$, so the basis

$\{(-1,1,0),(-1,0,1)\}$ gives dimension **2**.

d. A symmetric matrix is $\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so the dimension is **3**.

Answer to Question 5

a. $T(\lambda\mathbf{u}+\mu\mathbf{v})$ has components $\lambda(u_1+u_2)+\mu(v_1+v_2)$ and $\lambda(2u_1-u_2)+\mu(2v_1-v_2)$, which equals $\lambda T(\mathbf{u})+\mu T(\mathbf{v})$. The matrix columns are $T(1,0)=(1,2)$ and $T(0,1)=(1,-1)$, so the matrix is $\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.

b. $\det = -1-2 = -3 \neq 0$, so T is **invertible**.

c. $S(0,0)=(1,0)\neq(0,0)$. A linear map must send $\mathbf{0}$ to $\mathbf{0}$, so S is **not linear** (it is a translation).

Answer to Question 6

a. Solve $A\mathbf{x}=\mathbf{0}$. Row 2 is twice row 1, so we need $x+2y+3z=0$ and $x+y+z=0$. Subtracting: $y+2z=0$, so $y=-2z$ and $x=z$. The kernel is spanned by $(1,-2,1)$ (**nullity 1**).

b. The image is the span of the columns $(1,2,1)$, $(2,4,1)$ and $(3,6,1)$. The third column is $(3,6,1)=2(2,4,1)-(1,2,1)$, so a basis is $\{(1,2,1),(2,4,1)\}$ (**rank 2**).

c. $\text{rank}+\text{nullity}=2+1=3=\dim\mathbb{R}^3$.

Answer to Question 7

a. The determinant of the matrix with columns $\mathbf{p}_i$ is $2(1-0)-0+1(1-0)=3\neq 0$, so they are independent.

b. Solve $2a+c=5$, $a+b=4$, $b+c=3$. From the second, $b=4-a$; from the third $c=3-b=a-1$; first: $2a+a-1=5$ gives $a=2$, so $b=2$, $c=1$. The recipe is **2 units of $\mathbf{p}_1$, 2 of $\mathbf{p}_2$, 1 of $\mathbf{p}_3$**.

c. Three independent vectors in $\mathbb{R}^3$ form a basis, so every vector has exactly one coordinate triple.

d. For instance $(0,0,1)$: solve $2a+c=0$, $a+b=0$, $b+c=1$: $c=-2a$, $b=-a$, so $-a-2a=1$ and $a=-\frac{1}{3}$, which is negative. Such colours are outside the range the three inks can produce.

Answer to Question 8

a. The determinant is [redacted], so the vectors are independent and form a basis.
b. Solve $a + c = 3$, $2a + b = 7$, $2b + c = 7$. From the first $c = 3 - a$; second $b = 7 - 2a$; third: $14 - 4a + 3 - a = 7$, so $a = 2$, $b = 3$, $c = 1$. Servings: **2, 3 and 1**.
c. Two independent vectors span a plane through the origin in $\mathbb{R}^3$.
d. Only if $(3, 7, 7)$ lies in that plane. By (b) the coordinates are unique, and the coefficient of $\mathbf{f}_3$ is $1 \neq 0$, so the answer is **no**.

Answer to Question 9

a. The image (column space) has dimension equal to the rank: **3**.
b. $\text{rank} + \text{nullity} = 6$ (the number of columns), so the nullity is $6 - 3 = \mathbf{3}$.
c. Three independent changes of the readings leave all four summaries unchanged, so they cannot be detected from the summaries.
d. No. The image has dimension 3, but $\mathbb{R}^4$ has dimension 4, so the summaries always lie in a three-dimensional subspace.

Answer to Question 10

a. $D(1) = 0$, $D(t) = 1$, $D(t^2) = 2t$. In the basis $\{1, t, t^2\}$ the columns are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$: [redacted].
b. The kernel is the constant polynomials (dimension 1). The image is $\{a + bt\}$, spanned by 1 and t (dimension 2). Check: $1 + 2 = 3 = \dim P_2$.
c. The coordinate vector of h is $(2, 20, -4.9)$. Then [redacted], that is $D(h) = 20 - 9.8t$.
d. $D(h)$ is the velocity of the ball. $D^2h = -9.8$ is the (constant) acceleration due to gravity.

Answer to Question 11

a. $\begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$, since $S(1, 0) = (1, 0)$ and $S(0, 1) = (0.5, 1)$.
b. $(0, 0) \mapsto (0, 0)$; $(1, 0) \mapsto (1, 0)$; $(0, 2) \mapsto (1, 2)$; $(1, 2) \mapsto (2, 2)$. The rectangle becomes a parallelogram with the top edge shifted one unit to the right.
c. $\det S = 1$: the area of every shape is unchanged by the shear, so letters keep the

same ink area.

d. $S^{-1} = \begin{pmatrix} 1 & -0.5 \\ 0 & 1 \end{pmatrix}$, which shears back by the same amount in the opposite direction.

Answer to Question 12

a. The determinant of the matrix with rows $(1, 1)$, $(1, -1)$ is $-2 \neq 0$, so the vectors are independent; two independent vectors in $\mathbb{R}^2$ form a basis.
b. Solve $a + b = 3$, $a - b = 1$: $a = 2$, $b = 1$. Coordinates $(2, 1)$.
c. $a + b = 2$, $a - b = 6$: $a = 4$, $b = -2$. Coordinates $(4, -2)$.
d. The reflection fixes $\mathbf{d}_1 = (1, 1)$ and sends $\mathbf{d}_2 = (1, -1)$ to $(-1, 1) = -\mathbf{d}_2$. In the basis $\mathcal{B}$ its matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Answer to Question 13

a. $R(1, 0) = (0, 1)$ and $R(0, 1) = (-1, 0)$, so the matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
b. No non-zero vector is sent to zero, so the kernel is $\{\mathbf{0}\}$. By rank-nullity the rank is 2, so the image is all of $\mathbb{R}^2$. Hence R is injective and surjective: an **isomorphism**.
c. After 1 minute: $(-1, 3)$. After 2: $(-3, -1)$. After 3: $(1, -3)$. After 4: $(3, 1)$.
d. Four quarter turns make one full turn, which returns every vector to its starting position. In matrices, $R^2 = -I$, so $R^4 = (-I)^2 = I$.

Answer to Question 14

a. Carbon: $3a = c$. Hydrogen: $8a = 2d$. Oxygen: $2b = 2c + d$.
b. $A =$ [redacted]. From the first two equations $c = 3a$ and $d = 4a$. The third gives $2b = 6a + 4a = 10a$, so $b = 5a$. The kernel is spanned by $(1, 5, 3, 4)$.
c. The kernel has dimension 1. There are four unknowns and three independent equations, so $4 - 3 = 1$ by rank-nullity; the single free parameter is the overall scale of the reaction.
d. Taking $a = 1$: $\mathrm{C_3H_8} + 5\,\mathrm{O_2} \to 3\,\mathrm{CO_2} + 4\,\mathrm{H_2O}$.

Answer to Question 15

a. U: $\{(-1, 1, 0), (-1, 0, 1)\}$ (dimension 2). W: $(x, x, z) = x(1, 1, 0) + z(0, 0, 1)$, basis $\{(1, 1, 0), (0, 0, 1)\}$ (dimension 2).

b. Points with $x = y$ and $x + y + z = 0$ have $z = -2x$: $(x, x, -2x) = x(1, 1, -2)$. So $U \cap W$ is the line spanned by $(1, 1, -2)$, dimension **1**.
c. $\dim(U + W) = 2 + 2 - 1 = \mathbf{3}$, so $U + W = \mathbb{R}^3$.
d. $(1, -1, 0) \in U$ and $(0, 0, 1) \in W$, but their sum $(1, -1, 1)$ has $x + y + z = 1 \neq 0$ and $x \neq y$, so it is in neither: $U \cup W$ is not closed under addition.

Answer to Question 16

a. $T(\lambda p + \mu q)$ has entries $\lambda p(t) + \mu q(t)$, so T is linear. The images are $T(1) = (1, 1, 1)$, $T(t) = (0, 1, 2)$, $T(t^2) = (0, 1, 4)$, so the matrix is [redacted].
b. $\det = 1(4 - 2) = 2 \neq 0$, so T is invertible: for any three values there is exactly one polynomial of degree at most 2.
c. Solve $a = 1$, $a + b + c = 3$, $a + 2b + 4c = 7$: $b + c = 2$, $2b + 4c = 6$ so $c = 1$, $b = 1$. Hence $p(t) = 1 + t + t^2$.
d. $p(3) = 1 + 3 + 9 = 13$. Three data points fit a quadratic exactly, but a perfect fit does not mean the prediction is reliable: seasonal or random effects are ignored.

Answer to Question 17

a. Row 1 + row 2 $= (1, 3, 1, 2) =$ row 3. The first two rows are independent, so the rank is **2**.
b. Solve $x_1 + 2x_2 + x_4 = 0$ and $x_2 + x_3 + x_4 = 0$. Put $x_3 = s$, $x_4 = t$: $x_2 = -s - t$ and $x_1 = -2x_2 - x_4 = 2s + 2t - t = 2s + t$. So $(x_1, \ldots, x_4) = s(2, -1, 1, 0) + t(1, -1, 0, 1)$. A basis: $\{(2, -1, 1, 0), (1, -1, 0, 1)\}$.
c. The columns are $(1, 0, 1)$, $(2, 1, 3)$, $(0, 1, 1)$, $(1, 1, 2)$. The first two are independent and the others are combinations of them, so a basis of the image is $\{(1, 0, 1), (2, 1, 3)\}$.
d. Rank 2 + nullity $2 = 4$, the number of columns. The kernel consists of signal patterns that the monitors cannot detect.

Answer to Question 18

a. $t = 0$: $a + b = 0$. $t = \frac{\pi}{2}$: $a + c = 0$. $t = \pi$: $a - b = 0$. From the first and third, $a = b = 0$ (adding gives $2a = 0$), then $c = 0$.
b. The only linear combination that equals the zero function is the trivial one, so the three functions are **linearly independent**.
c. $g = 1 \cdot f_2 + 1 \cdot f_3$, so the coordinates with respect to (f_2, f_3) are $(1, 1)$.
d. $\{\cos t, \sin t\}$ is a basis: both solve $y'' + y = 0$, and they are independent. The constant function 1 does not solve it.

Answer to Question 19

a. $T(1,1) = (4,4) = 4(1,1)$ and $T(1,-1) = (2,-2) = 2(1,-1)$.

b. The matrix is diagonal: $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$.

c. $(2,0) = 1(1,1) + 1(1,-1)$. Applying T^5 multiplies the first component by $4^5 = 1024$ and the second by $2^5 = 32$:
$T^5(2,0) = 1024(1,1) + 32(1,-1) = (1056, 992)$.

d. The direction $(1,1)$ (equal amounts in each tank) is multiplied by 4 each step, while the direction $(1,-1)$ (difference between the tanks) is multiplied by 2. The total grows faster than the imbalance.

Answer to Question 20

a. $(\alpha+\gamma)\mathbf{u} + (\alpha+\beta)\mathbf{v} + (\beta+\gamma)\mathbf{w} = \mathbf{0}$.

b. By independence all three coefficients vanish: $\alpha+\gamma = 0$, $\alpha+\beta = 0$, $\beta+\gamma = 0$.

c. Adding all three: $2(\alpha+\beta+\gamma) = 0$, so $\alpha+\beta+\gamma = 0$. Subtracting each equation: $\beta = 0$, $\gamma = 0$, $\alpha = 0$. Hence only the trivial combination works.

d. The matrix is  with determinant 2. A non-zero determinant means the new vectors are independent. Modulo 2 the determinant is 0 (and in fact $(\mathbf{u}+\mathbf{v}) + (\mathbf{v}+\mathbf{w}) + (\mathbf{w}+\mathbf{u}) = 2(\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0}$), so the vectors are dependent: the step "divide by 2" is impossible.