Worksheet - Statistical inference

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Sample mean, variance and standard error *(Section A · Skills practice)*

A random sample of 8 values is: 12, 15, 11, 18, 14, 16, 13, 17.

a. Find the sample mean $\bar{x}$. [1 mark]
b. Find the sample variance $s^2 = \dfrac{1}{n-1}\sum(x_i - \bar{x})^2$ and the standard deviation s. [3 marks]
c. Find the estimated standard error of the mean. [2 marks]
d. Explain why we divide by $n-1$ and not by n. [2 marks]

Question 2

Question 2 – The sampling distribution of the mean *(Section A · Skills practice)*

A population has mean $\mu = 40$ and standard deviation $\sigma = 10$. Samples of size $n = 25$ are taken.

a. State the approximate distribution of $\bar{X}$ and justify it. [2 marks]
b. Find $P(\bar{X} > 43)$. [2 marks]
c. Find $P(38$. [2 marks]
d. How large must n be for the standard error to be at most 1? [2 marks]

Question 3

Question 3 – Confidence intervals *(Section A · Skills practice)*

Find a 95% confidence interval in each case.

a. $\sigma = 6$ known, $n = 36$, $\bar{x} = 72.4$. [2 marks]
b. A normal population, $n = 10$, $\bar{x} = 15.2$, $s = 2.1$. [3 marks]
c. 84 successes in a sample of 200 (a proportion). [3 marks]

Question 4

Question 4 – Tests for a mean *(Section A · Skills practice)*

Carry out each test, stating the hypotheses and the conclusion.

a. $H_0 : \mu = 500$ against $H_1 : \mu \neq 500$ with $\sigma = 20$, $n = 16$, $\bar{x} = 509$, at the 5% level. [3 marks]
b. The same data with $H_1 : \mu > 500$. [2 marks]
c. A normal sample with $n = 12$, $\bar{x} = 4.6$, $s = 0.9$: test $H_0 : \mu = 4$ against $H_1 : \mu \neq 4$ at 5%. [3 marks]

Question 5

Question 5 – A chi-squared goodness-of-fit test *(Section A · Skills practice)*

A die is rolled 120 times.

Score	1	2	3	4	5	6
Observed	25	17	15	23	24	16

a. State the hypotheses and the expected frequencies. [2 marks]
b. Calculate $\chi^2 = \sum \dfrac{(O-E)^2}{E}$. [3 marks]
c. Test at the 5% level. [2 marks]

Question 6

Question 6 – Correlation and the regression line *(Section A · Skills practice)*

Five pairs of data are given.

x	1	2	3	4	5
y	3	5	6	8	9

a. Find S_{xx}, S_{xy} and S_{yy}. [3 marks]
b. Find the least-squares line $y = a + bx$. [2 marks]
c. Find the correlation coefficient r and comment. [2 marks]

Question 7

Question 7 – Battery life of a new phone *(Section B · Real-life practice)*

A consumer magazine tests 40 phones of a new model. The mean battery life is $\bar{x} = 18.6$ hours with standard deviation $s = 2.5$ hours. The manufacturer advertises "20 hours of battery".

a. Find a 95% confidence interval for the mean battery life.
b. Does the interval support the advert? Explain.
c. Find a 99% interval and explain why it is wider.
d. How many phones should be tested to estimate the mean to within ± 0.5 hours with 95% confidence?

Question 8

Question 8 – Is the coffee machine giving short measures? *(Section B · Real-life practice)*

A café's coffee machine is set to pour 250 ml. The volume poured is normal with $\sigma = 8$ ml. The manager measures 25 cups and finds a mean of 246.5 ml.

a. State suitable hypotheses.
b. Calculate the test statistic and the p-value.
c. Give conclusions at the 5% and at the 1% levels.
d. Explain in context what a Type I error would be.

Question 9

Question 9 – An opinion poll before an election *(Section B · Real-life practice)*

In a random sample of 1 000 voters, 520 say they will vote for the Green candidate. A newspaper headline says "Green candidate set to win a majority".

a. Find a 95% confidence interval for the proportion p of supporters.
b. Test $H_0 : p = 0.5$ against $H_1 : p > 0.5$ at the 5% level.
c. Is the headline justified?
d. What sample size would give a margin of error of 2 percentage points?

Question 10

Question 10 – Does the revision app work? *(Section B · Real-life practice)*

Eight students take a test, use a revision app for two weeks, then take a similar test. The improvements $d =$ after $-$ before are shown.

Student	A	B	C	D	E	F	G	H
d	5	3	-1	6	4	2	7	2

a. Explain why a paired test is appropriate.
b. Find $\bar{d}$ and s_d.
c. Test at the 5% level whether the mean improvement is positive.
d. Suggest a weakness of this study design.

Question 11

Question 11 – Comparing two production lines *(Section B · Real-life practice)*

Two production lines make the same component. Samples give the following lengths (in mm).

Line	n	Mean	Standard deviation

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1	50	12.4	1.5
2	60	11.8	1.8

a. State hypotheses for a test of whether the mean lengths differ.
b. Calculate the standard error of $\bar{X}_1 - \bar{X}_2$.
c. Calculate the test statistic and the p-value.
d. State your conclusion at the 5% and 10% levels.

Question 12

Question 12 – Road accidents during the week *(Section B · Real-life practice)*

A council records 130 road accidents on weekdays over a year.

Day	Mon	Tue	Wed	Thu	Fri
Accidents	30	22	18	25	35

a. State hypotheses for a test of whether accidents are equally likely on each weekday.
b. Calculate the expected frequencies and χ^2.
c. Carry out the test at the 5% level.
d. Which day contributes most to χ^2, and what might explain it?

Question 13

Question 13 – Ice cream sales and temperature *(Section B · Real-life practice)*

An ice cream van records the midday temperature and the number of ice creams sold on six summer days.

Temperature x (°C)	14	16	20	23	26	29
Sales y	120	140	175	200	240	260

a. Given $S_{xx} = 167.33$ and $S_{xy} = 1586.67$, find the regression line of y on x.
b. Interpret the gradient in context.
c. Predict the sales at 25 °C.
d. Given $S_{yy} = 15120.8$, find r.
e. Would you use the line to predict sales at 5 °C? Explain.

Question 14

Question 14 – Quality control and the power of a test *(Section B · Real-life practice)*

Bags of flour should contain 100 g on average; the mass is normal with $\sigma = 4$ g. An inspector weighs 16 bags and tests $H_0 : \mu = 100$ against $H_1 : \mu$ at the 5% level.

a. Show that H_0 is rejected when $\bar{x}$.
b. What is the probability of a Type I error?
c. If in fact $\mu = 97$, find the probability of a Type II error.
d. Find the power of the test when $\mu = 97$ and explain how it could be increased.

Question 15

Question 15 – Exercise and sleep quality *(Section C · Problem solving)*

A health survey of 170 adults records their level of exercise and the quality of their sleep.

	Low exercise	Medium	High	Total
Good sleep	20	35	45	100
Poor sleep	30	25	15	70
Total	50	60	60	170

a. State the hypotheses for a test of independence.
b. Show that the expected frequency for "good sleep, low exercise" is 29.41 and find the others.
c. Calculate χ^2.
d. Find the degrees of freedom and test at the 1% level.
e. A journalist writes "exercise improves sleep". Comment.

Question 16

Question 16 – The central limit theorem with dice *(Section C · Problem solving)*

A single fair die has mean $\mu = 3.5$ and variance $\sigma^2 = \dfrac{35}{12}$. A class repeatedly rolls 30 dice and records the mean score $\bar{X}$. Out of 200 samples, the results were:

$\bar{X}$	below 3	3 to 3.5	3.5 to 4	above 4
Frequency	11	91	86	12

a. Show that $\sigma^2 = \dfrac{35}{12}$.
b. State the approximate distribution of $\bar{X}$.
c. Find $P(\bar{X} > 4)$ and the expected number of samples above 4.
d. Compare with the class results.
e. Explain why the distribution of $\bar{X}$ is close to normal although a single die is uniform.

Question 17

Question 17 – Testing the strength of bridge cables *(Section C · Problem solving)*

An engineer tests six steel cables for a footbridge. The breaking loads (in kN) are below. The specification requires a mean breaking load above 50 kN. Assume the loads are normally distributed.

Cable	1	2	3	4	5	6
Load (kN)	52.1	49.8	51.5	50.6	48.9	51.3

a. Calculate $\bar{x}$ and s.
b. Find a 95% confidence interval for the mean breaking load.
c. Test $H_0 : \mu = 50$ against $H_1 : \mu > 50$ at the 5% level.
d. The engineer says "the cables are within specification". Is this justified? What would you recommend?

Question 18

Question 18 – Correlation is not causation *(Section C · Problem solving)*

Over 10 summer weeks at a seaside resort, the number of ice creams sold and the number of lifeguard rescues give a sample correlation coefficient $r = 0.72$.

a. To test $H_0 : \rho = 0$ the statistic $t = r\sqrt{\dfrac{n-2}{1-r^2}}$ has a t_{n-2} distribution. Calculate it.
b. Test at the 5% level (two-sided).
c. A local councillor suggests banning ice cream to reduce rescues. Explain the flaw.
d. Name the likely lurking variable and describe a better analysis.

Question 19

Question 19 – Planning a medical trial *(Section C · Problem solving)*

Researchers plan a study to detect whether a new diet lowers blood pressure by 2 mmHg on average. Individual changes have standard deviation $\sigma = 8$ mmHg. They will test $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ at the 5% level.

a. For a sample of size n, write the standard error and the rejection rule.
b. For $n = 100$, find the power of the test when $\mu = 2$ (ignore the very small chance of rejecting in the wrong tail).
c. Show that a power of 90% needs $n \geq \left(\dfrac{(1.96 + 1.2816) \times 8}{2}\right)^2$ and evaluate it.
d. Explain the trade-off between the significance level, the power and the cost of the trial.

Question 20

Question 20 – Challenge: why divide by n − 1? *(Section D · Challenge)*

Let $X_1, \ldots, X_n$ be independent with mean μ and variance σ^2, and let $\bar{X} = \frac{1}{n}\sum X_i$.

a. Show that $E[\bar{X}] = \mu$ and $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$.

b. Show that $\sum (X_i - \bar{X})^2 = \sum X_i^2 - n\bar{X}^2$.

c. Using $E[X_i^2] = \sigma^2 + \mu^2$ and $E[\bar{X}^2] = \dfrac{\sigma^2}{n} + \mu^2$, show that $E\left[\sum (X_i - \bar{X})^2\right] = (n-1)\sigma^2$.

d. Deduce that $S^2 = \dfrac{1}{n-1}\sum (X_i - \bar{X})^2$ is unbiased, and find the bias of the version with $\frac{1}{n}$.