Test answers - Sequences and series convergence

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Dividing top and bottom by n, $a_n = \dfrac{2 - 1/n}{1 + 1/n} \to 2$, so $L = 2$ (1). Then $a_n - 2 = \dfrac{2n - 1 - 2n - 2}{n + 1} = \dfrac{-3}{n + 1}$, so $|a_n - L| = \dfrac{3}{n + 1}$ (1).

b. Let $\varepsilon > 0$ be given and choose an integer $N \geq 3/\varepsilon$ (1). For every $n > N$, $|a_n - 2| = \dfrac{3}{n + 1} < \dfrac{3}{N} \leq \varepsilon$ (1). Such an N exists for every ε, so $a_n \to 2$ by definition.

In (b), N depends on ε: for example, $\varepsilon = 0.01$ needs $N \geq 300$, and indeed $|a_{301} - 2| = \frac{3}{302} < 0.01$. Note that $\frac{3}{n+1} < \frac{3}{N}$ because $n + 1 > N$. The sequence is increasing, since $a_n = 2 - \frac{3}{n+1}$.

Answer to Question 2 (4 marks)

a. By partial fractions, $\dfrac{4}{(2n - 1)(2n + 1)} = \dfrac{2}{2n - 1} - \dfrac{2}{2n + 1}$ (1). In the partial sum each negative term cancels the next positive one, leaving $2 - \dfrac{2}{2N + 1}$, which tends to 2 (1).

b. This is a geometric series with first term 3 and ratio $r = -\frac{1}{2}$; since $|r| < 1$ it converges (1). Its sum is $\dfrac{3}{1 - (-1/2)} = \dfrac{3}{1 + 1/2} = 2$ (1).

In (a), this is a telescoping series; check the first partial sum, $\frac{4}{1\times 3} = \frac{4}{3} = 2 - \frac{2}{3}$. In (b), the partial sums 3, 1.5, 2.25, 1.875 oscillate towards 2.

Answer to Question 3 (4 marks)

a. Since $n^3 + 1 > n^3$, we have $0 < \dfrac{n}{n^3 + 1} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$ (1). The series $\sum \frac{1}{n^2}$ converges ($p = 2 > 1$), so the given series converges by the comparison test (1).

b. Use the ratio test: $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n + 1)^2} \cdot \dfrac{n^2}{2^n} = 2\left(\dfrac{n}{n + 1}\right)^2 \to 2 > 1$ (1), so the series diverges (1). In fact the terms tend to infinity.

In (a), the limit comparison test with $\frac{1}{n^2}$ also works, since $\frac{n}{n^3+1} \div \frac{1}{n^2} \to 1$. In (b), the divergence test alone is enough, because $\frac{2^n}{n^2}$ does not tend to zero.

Answer to Question 4 (4 marks)

a. The series of absolute values is $\sum \frac{1}{n^2}$, a p-series with $p = 2 > 1$, which converges. So the series converges absolutely (2), and absolute convergence implies convergence.
b. The first four terms give $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \approx 0.7986$ (1). The series alternates with terms decreasing to 0, so the error is less than the next term, $\frac{1}{25} = 0.04$ (1). The true value $\frac{\pi^2}{12} \approx 0.8225$ confirms this.

The actual error is about 0.024, well within the bound of 0.04.

Answer to Question 5 (4 marks)

a. Apply the ratio test to the absolute values:



(1). The series converges when this is less than 1, that is $|x - 1| < 2$, so $R = 2$ (1), centred at $x = 1$.
b. The ratio test is inconclusive at the endpoints, so test them directly. At $x = 3$ the series is $\sum \frac{1}{n}$, the harmonic series, which diverges (1). At $x = -1$ it is $\sum \frac{(-1)^n}{n}$, which converges by the alternating series test. The interval of convergence is $[-1, 3)$ (1).