# Lesson - Sequences and series convergence

Year: University Year 1

Analysis asks when an infinite process settles down. This chapter covers limits of sequences, convergence tests for series and the radius of convergence of power series.

### Starter activity

Fold a sheet of paper in half and tear it; keep one half and tear it in half again, and so on. Lay the pieces next to each other. What fraction of the sheet do the first five pieces make? Can the pieces ever make more than one sheet? Now compare with the sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$: do you think it stays bounded too?

## Sequences and their limits

### Definition: convergent sequence

A **sequence** $(a_n)$ is a function from the positive integers to $\mathbb{R}$. It **converges** to $L$ if for every $\varepsilon > 0$ there is an $N$ such that $n > N \Rightarrow |a_n - L|$. It is **monotonic** if it is always non-decreasing or always non-increasing.

### Theorem: monotone convergence theorem

A bounded monotonic sequence converges. Limits of sequences obey the same algebra as limits of functions.

### Example 1

**Show that $a_n = \frac{2n+1}{n+3} \to 2$.**

$|a_n - 2| = \frac{5}{n+3}$ as soon as $n > \frac{5}{\varepsilon} - 3$, so choose $N \geq 5/\varepsilon$. Also $\frac{3n^2+1}{n^2+n} \to 3$ by dividing through by $n^2$.

### Example 2: recursive sequence

**Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Find the limit.**

By induction $a_n$ and $a_{n+1} > a_n$, so $a_n$ is bounded and increasing, and converges to some $L$ with $L = \sqrt{2 + L}$, that is $L^2 - L - 2 = 0$, so $L = 2$.

## Series

**Definition: convergent series**

The series $\sum_{n=1}^{\infty} a_n$ converges if its partial sums $s_N = \sum_{n=1}^{N} a_n$ converge.

**Rule: divergence test**

If $a_n \not\to 0$ then $\sum a_n$ diverges. The converse is false: the harmonic series $\sum \frac{1}{n}$ diverges since $s_{2^k} \geq 1 + \frac{k}{2}$.

**Property: geometric and telescoping series**

$\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ for $|r|$ (diverges otherwise). Telescoping: $\sum \frac{1}{n(n+1)} = \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$.

**Example 3**

**Find $\sum_{n=0}^{\infty} 3(\frac{1}{2})^n$.**

$a = 3$, $r = \frac{1}{2}$, so the sum is $\frac{3}{1-1/2} = 6$.

## Convergence tests for positive terms

**Rule: comparison tests**

**p-series:** $\sum \frac{1}{n^p}$ converges if and only if $p > 1$. **Comparison test:** if $0 \leq a_n \leq b_n$ and $\sum b_n$ converges then $\sum a_n$ converges (and if $\sum a_n$ diverges so does $\sum b_n$). **Limit comparison:** if $a_n / b_n \to c \in (0, \infty)$, both series behave alike.

**Rule: ratio and root tests**

If $\left|\dfrac{a_{n+1}}{a_n}\right| \to \rho$ (ratio test), or $|a_n|^{1/n} \to \rho$ (root test), the series converges absolutely for $\rho$, diverges for $\rho > 1$, and the test is inconclusive for $\rho = 1$.

**Method: choosing a test**

1. If $a_n \not\to 0$, the series diverges.
2. For factorials and powers, use the ratio test.
3. For rational functions, use comparison or limit comparison with a p-series.
4. If the signs alternate, use the alternating series test.

**Example 4**

**Test $\sum \frac{1}{n^2+1}$, $\sum \frac{2n+1}{n^3+5}$, $\sum \frac{n}{2^n}$, $\sum \frac{n!}{n^n}$ and $\sum \left(\frac{n}{2n+1}\right)^n$.**

The first converges by comparison with $\sum \frac{1}{n^2}$. The second converges (compare $\frac{2}{n^2}$). The third has $\rho = \frac{1}{2}$ and converges (its sum is 2). The fourth has $\rho = \frac{1}{e}$ and converges. The fifth has root limit $\frac{1}{2}$ and converges.

## Alternating and conditional convergence

**Theorem: alternating series test**

If $b_n \geq 0$ is decreasing with $b_n \to 0$ then $\sum (-1)^{n+1} b_n$ converges.

**Definition: absolute and conditional convergence**

A series is **absolutely convergent** if $\sum |a_n|$ converges (this implies convergence), and **conditionally convergent** if it converges but $\sum |a_n|$ diverges.

**Example 5**

**Classify $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ and $\sum \frac{(-1)^n}{n^2}$.**

The first converges (to $\ln 2$) by the alternating series test, but $\sum \frac{1}{n}$ diverges, so it is conditionally convergent. The second is absolutely convergent, since $\sum \frac{1}{n^2}$ converges.

## Power series and radius of convergence

**Definition: radius of convergence**

A **power series** $\sum c_n (x-a)^n$ has a **radius of convergence** $R \in [0, \infty]$: it converges absolutely for $|x-a|$ and diverges for $|x-a| > R$. Endpoints must be checked separately. By the ratio test, .

**Property: term-by-term calculus**

Inside the interval of convergence, power series can be differentiated and integrated term by term.

**Example 6**

**Find the interval of convergence of $\sum \frac{x^n}{n}$, $\sum \frac{x^n}{n!}$ and $\sum \frac{(x-2)^n}{3^n n}$.**

For $\sum \frac{x^n}{n}$, $R = 1$; at $x = 1$ it diverges (harmonic), at $x = -1$ it converges (alternating),

so the interval is $[-1, 1)$. For $\sum \frac{x^n}{n!}$, $R = \infty$. For $\sum \frac{(x-2)^n}{3^n n}$, $R = 3$ and the interval is $[-1, 5)$. Integrating $\frac{1}{1-x} = \sum x^n$ gives $-\ln(1-x) = \sum \frac{x^n}{n}$, hence $\sum \frac{1}{n 2^n} = \ln 2$.

**Watch out**

- Concluding convergence from $a_n \to 0$.
- Using the ratio test when $\rho = 1$.
- Forgetting to test the endpoints of a power series.

## Working with infinite sums

**Key fact: the integral test**

Let $f$ be positive and decreasing on $[1, \infty)$ with $a_n = f(n)$. Then $\sum a_n$ and $\int_1^\infty f(x)\,dx$ either both converge or both diverge. The test also gives estimates: for the remainder after $N$ terms, $\int_{N+1}^\infty f(x)\,dx \le \sum_{n>N} a_n \le \int_N^\infty f(x)\,dx$. Applied to $f(x) = x^{-p}$ it proves the key result that $\sum \frac{1}{n^p}$ converges exactly when $p > 1$, which is the family most often used in comparison tests.

**Method: deciding convergence step by step**

1. Check that $a_n \to 0$. If not, the series diverges and you can stop.
2. Recognise a standard series: geometric, $p$-series, telescoping.
3. For factorials or powers like $c^n$, try the ratio test; for $n$th powers, the root test.
4. For rational functions of $n$, use limit comparison with $\frac{1}{n^p}$, where $p$ is the difference of the degrees.
5. For signs that alternate, use the alternating series test and then check absolute convergence separately.
6. State the test and the conclusion in words.

**Worked example 7: A sequence defined by a recurrence**

**Let $a_1 = 1$ and $a_{n+1} = \dfrac{a_n + 6}{2}$. Prove that $(a_n)$ converges and find its limit.**

The first terms are $1,\ 3.5,\ 4.75,\ 5.375$: they seem to increase towards 6. Induction: if $a_n$ then $a_{n+1}$, so the sequence is bounded above by 6. Also $a_{n+1} - a_n = \frac{6 - a_n}{2} > 0$, so it is increasing. By the monotone convergence theorem it has a limit $L$, and taking limits in the recurrence gives $L = \frac{L+6}{2}$, so $L = 6$.

**Worked example 8: Limit comparison in action**

**Decide whether $\displaystyle\sum_{n\ge 1}\frac{3n^2+1}{n^4+n}$ converges.**

For large $n$ the term behaves like $\dfrac{3n^2}{n^4}=\dfrac{3}{n^2}$. Compare with $b_n=\frac{1}{n^2}$: $\dfrac{a_n}{b_n}=\dfrac{3n^4+n^2}{n^4+n}\to 3$, a finite positive limit. Since $\sum\frac{1}{n^2}$ converges, so does the given series. The method only needs the leading powers of the numerator and denominator.

## Worked example 9: The ratio test with factorials

**Decide whether $\displaystyle\sum\frac{(2n)!}{(n!)^2\,5^n}$ converges.**

$\dfrac{a_{n+1}}{a_n}=\dfrac{(2n+2)(2n+1)}{(n+1)^2}\cdot\dfrac{1}{5}=\dfrac{2(2n+1)}{5(n+1)}\to\dfrac{4}{5}$. The series converges. The binomial coefficient $\binom{2n}{n}$ grows roughly like $4^n$, and $4^n/5^n$ tends to zero quickly, which explains the result.

## Worked example 10: Error in an alternating series

**How many terms of $\displaystyle\sum_{n\ge 1}\frac{(-1)^{n+1}}{n^3}$ are needed for an error below $0.001$?**

The terms $\frac{1}{n^3}$ decrease to zero, so the error after $N$ terms is at most $\frac{1}{(N+1)^3}$. We need $(N+1)^3>1000$, i.e. $N+1>10$, so $N=10$ terms suffice. The sum of ten terms is $0.90112$, and the exact value is $\frac{3}{4}\zeta(3)\approx 0.90154$.

## Worked example 11: Finding an interval of convergence

**Find the interval of convergence of $\displaystyle\sum_{n\ge 1}\frac{x^n}{n}$.**

The ratio test gives [redacted], so $R=1$. At $x=1$ we get the harmonic series, which diverges. At $x=-1$ we get the alternating harmonic series, which converges. The interval of convergence is $[-1,1)$. Always test the endpoints separately: the ratio test says nothing there.

## Worked example 12: A geometric series in context

**A charity receives £1 000 this year and expects each later year's gift to be 5% smaller than the previous one. What is the total of all future gifts, and after how many years has it received 90% of this total?**

The gifts form a geometric sequence with first term $1000$ and ratio $0.95$, so the total is

$\dfrac{1000}{1-0.95} = £20\,000$. After $n$ years the amount received is $20\,000(1-0.95^n)$, and this reaches 90% of the total when $0.95^n \le 0.1$, that is $n \ge \dfrac{\ln 10}{\ln(1/0.95)} \approx 44.9$. So it takes 45 years. A small ratio close to 1 produces a large total, but the total is reached very slowly.

## Key fact: Taylor series of standard functions

The most useful power series should be known by heart, together with where they are valid. For every real $x$: $e^x = \sum \frac{x^n}{n!}$, $\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$. For $|x|$ : $\frac{1}{1-x} = \sum x^n$, $\ln(1+x) = \sum_{n\ge 1} \frac{(-1)^{n+1} x^n}{n}$ (also at $x=1$), and $(1+x)^\alpha = \sum \binom{\alpha}{n} x^n$. New series are obtained from these by substitution, multiplication by a power of $x$, differentiation and integration, which is usually much faster than computing derivatives at a point.

## Worked example 13: Comparing growth rates

**Order the sequences $n^{10}$, $2^n$, $n!$ and $\ln n$ by how fast they grow, and use the result to find $\lim \dfrac{n^{10}}{2^n}$.**

For large $n$: $\ln n \ll n^{10} \ll 2^n \ll n!$. To justify the middle step, the ratio of successive terms of $a_n = \dfrac{n^{10}}{2^n}$ is $\dfrac{1}{2}\left(1+\frac{1}{n}\right)^{10} \to \dfrac{1}{2}$, so eventually each term is less than $0.6$ times the previous one, and $a_n \to 0$. This ordering of growth rates is used constantly when choosing a comparison series or predicting the result of a ratio test.

## Note: why convergence matters

Infinite series are everywhere in applied mathematics. Calculators evaluate $\sin$, $\exp$ and $\ln$ with truncated power series; engineers approximate integrals that have no closed form; physicists expand solutions in series when exact formulas are out of reach. In each case two questions must be answered: does the series converge, and how many terms are needed for a given accuracy? The tests of this chapter answer the first question, and the integral and alternating series bounds answer the second.

## Watch out: typical mistakes

- Concluding that a series converges because $a_n \to 0$: the harmonic series is the counterexample.
- Using the ratio test when the limit is 1: the test then gives no information.
- Forgetting to test the endpoints of the interval of convergence.
- Rearranging or regrouping the terms of a series that is only conditionally convergent.

**Did you know?**

Nicole Oresme proved around 1350 that the harmonic series diverges, long before calculus. Leonhard Euler found in 1734 that the sum of $\frac{1}{n^2}$ is $\frac{\pi^2}{6}$, solving the Basel problem that had defeated the Bernoulli brothers. The rigorous definition of convergence with $\varepsilon$ and $N$ came only in the nineteenth century, through the work of Cauchy and Weierstrass.

**Otto's tip**

Try the ratio test for factorials and powers, comparison for rational functions, and the alternating test for signs that flip.