Worksheet - Sequences and series convergence

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Limits of sequences *(Section A · Skills practice)*

Find the limit of each sequence as $n \to \infty$, showing your method.

a. $a_n = \dfrac{5n^2 - 3n}{2n^2 + 7}$ [2 marks]

b. $b_n = \sqrt{n^2 + n} - n$ [3 marks]

c. $c_n = \left(1 + \dfrac{2}{n}\right)^n$ [2 marks]

d. $d_n = n^{1/n}$ [3 marks]

Question 2

Question 2 – The epsilon definition *(Section A · Skills practice)*

Let $a_n = \dfrac{3n + 1}{n + 2}$.

a. Show that $|a_n - 3| = \dfrac{5}{n + 2}$. [2 marks]

b. Find the smallest N such that $|a_n - 3|$ for all $n > N$. [2 marks]

c. Prove from the definition that $a_n \to 3$. [3 marks]

Question 3

Question 3 – Geometric and telescoping series *(Section A · Skills practice)*

Find the sum of each series, or explain why it diverges.

a. $\displaystyle\sum_{n=0}^{\infty} 5\left(\tfrac{2}{3}\right)^n$ [2 marks]

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ [3 marks]

c. $\displaystyle\sum_{n=1}^{\infty} (0.9)^n$ [2 marks]

d. Write as a fraction. [2 marks]

Question 4

Question 4 – Comparison tests *(Section A · Skills practice)*

Decide whether each series converges, naming the test and the series you compare with.

a. $\sum \dfrac{1}{n^2+3n}$ [2 marks]

b. $\sum \dfrac{1}{\sqrt{n}+1}$ [2 marks]

c. $\sum \dfrac{n+1}{n^3+2}$ [2 marks]

d. $\sum \dfrac{\ln n}{n}$ [2 marks]

Question 5

Question 5 – Ratio and root tests *(Section A · Skills practice)*

Use the ratio test or the root test.

a. $\sum \dfrac{3^n}{n!}$ [2 marks]

b. $\sum \dfrac{n^2}{2^n}$ [2 marks]

c. $\sum \dfrac{n!}{2^n}$ [2 marks]

d. $\sum \left(\dfrac{2n}{3n+1}\right)^n$ [2 marks]

Question 6

Question 6 – Alternating series and power series *(Section A · Skills practice)*

Answer each part.

a. Show that $\sum \dfrac{(-1)^n}{\sqrt{n}}$ converges but not absolutely. [3 marks]

b. Is $\sum \dfrac{(-1)^n n}{n^2+1}$ absolutely convergent, conditionally convergent or divergent? [3 marks]

c. Find the radius and interval of convergence of $\sum_{n\ge 1} \dfrac{n\,x^n}{3^n}$. [4 marks]

Question 7

Question 7 – A bouncing ball *(Section B · Real-life practice)*

A ball is dropped from a height of 2 m. After each bounce it rises to $\frac{3}{4}$ of its previous height. A fall from height h takes $\sqrt{2h/g}$ seconds, with $g = 9.8$ m s^{-2}.

Bounce	0 (drop)	1	2	3
Height reached (m)	2	1.5	1.125	?

a. Complete the table.
b. Find the total distance travelled by the ball, assuming it bounces for ever.
c. Show that the times of successive falls form a geometric sequence with ratio $\sqrt{0.75}$.
d. Find the total time before the ball comes to rest.
e. Comment on how a series can have infinitely many terms but a finite total time.

Question 8

Question 8 – Repeated doses of a medicine *(Section B · Real-life practice)*

A patient takes 100 mg of a medicine every 12 hours. After 12 hours only 30% of the amount in the body remains. Let a_n be the amount just after the nth dose.

a. Show that $a_1 = 100$, $a_2 = 130$, $a_3 = 139$.
b. Show that $a_n = 100(1 + 0.3 + 0.3^2 + \cdots + 0.3^{n-1})$ and find a closed form.
c. Find a_5 and the long-term level $\lim a_n$.
d. The safe maximum level is 150 mg. Is the dose safe in the long run? What dose would give a long-term level of 120 mg?

Question 9

Question 9 – Saving every month *(Section B · Real-life practice)*

Priya pays £200 into a savings account at the end of every month. The account pays 0.4% interest per month, added at the end of each month before her payment.

a. Show that after 3 months she has $200(1.004^2 + 1.004 + 1)$.
b. Show that after n months the balance is $B_n = 200\dfrac{1.004^n - 1}{0.004}$.
c. Find the balance after 24 months and the interest earned.
d. How many months does it take for the balance to exceed £10 000?

Question 10

Question 10 – Achilles and the tortoise *(Section B · Real-life practice)*

Achilles runs at 10 m/s and a tortoise at 1 m/s. The tortoise starts 100 m ahead. Zeno argued: when Achilles reaches the tortoise's starting point, it has moved on; when he reaches that point, it has moved on again, and so on.

Stage	1	2	3	4
Time for Achilles (s)	10	1	0.1	?

a. Explain the times in the table and complete it.
b. Find the total time by summing the geometric series.
c. Check by solving $10t = 100 + t$ directly.
d. How far does Achilles run before he catches the tortoise?

Question 11

Question 11 – The Koch snowflake *(Section B · Real-life practice)*

Start with an equilateral triangle of side 1 and area A_0. At each stage, every side is divided into three and an outward equilateral triangle is built on the middle third.

Stage n	0	1	2
Number of sides	3	12	48
Side length	1	$\frac{1}{3}$	$\frac{1}{9}$

a. Show that the perimeter at stage n is $3\left(\frac{4}{3}\right)^n$ and that it tends to infinity.
b. At stage $n \geq 1$, $3 \times 4^{n-1}$ new triangles are added, each with area $A_0/9^n$. Write the total added area as a series.
c. Show that the area tends to $\frac{8}{5}A_0$.
d. Comment on a shape with infinite perimeter but finite area.

Question 12

Question 12 – Stacking books over the edge *(Section B · Real-life practice)*

Identical books of length 1 are stacked at the edge of a table. With the best arrangement, the top book can stick out by $\frac{1}{2}$, the next by $\frac{1}{4}$ more, then $\frac{1}{6}$, and so on: with n books the overhang is $\frac{1}{2}H_n$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

n	1	2	3	4
H_n	1	1.5	1.833	?

a. Complete the table.
b. How many books are needed for the top book to be completely beyond the table edge?
c. Explain why any overhang, however large, is possible in theory.
d. It takes 31 books to reach an overhang of 2 lengths. Check with $H_{30} \approx 3.995$ and $H_{31} \approx 4.027$.

Question 13

Question 13 – How a calculator finds e to the power 0.5 *(Section B · Real-life practice)*

A calculator evaluates $e^{0.5}$ using partial sums $S_N = \sum_{k=0}^{N} \frac{0.5^k}{k!}$.

N	2	4	6
S_N	1.625	1.6484375	?

a. Find S_6 to 7 decimal places.
b. Show that the tail after S_N is less than $\frac{0.5^{N+1}}{(N+1)!} \cdot \frac{1}{1-0.5}$.
c. Find the smallest N that guarantees an error below 10^{-6}.
d. Why does this series converge for every x?

Question 14

Question 14 – A fish population *(Section B · Real-life practice)*

A lake is restocked every year. Each year 20% of the fish die or are caught, and then 400 new fish are added: $a_{n+1} = 0.8a_n + 400$, with $a_0 = 500$.

a. Find a_1, a_2, a_3.
b. If the sequence converges to L, show that $L = 2000$.
c. Show that $a_n = 2000 - 1500(0.8)^n$.
d. Prove that (a_n) is increasing and bounded above by 2000.
e. After how many years does the population first exceed 1900?

Question 15

Question 15 – Estimating the Basel sum *(Section C · Problem solving)*

Euler proved in 1734 that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Before that, mathematicians could only estimate it. Let $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$ and S the full sum.

N	5	10	100
S_N	1.463611	1.549768	1.634984

a. Use the integral test to show $\int_{N+1}^{\infty} \frac{dx}{x^2}$.
b. Deduce that $S_N + \frac{1}{N+1}$.
c. Use $N = 10$ to find bounds for S.

d. Compare with $\pi^2/6$ and say how many terms would be needed to get the same accuracy from S_N alone.

Question 16

Question 16 – Two series for ln 2 *(Section C · Problem solving)*

Start from $\dfrac{1}{1-x} = \sum_{n\geq 0} x^n$ for $|x|$.

a. Integrate term by term to show $-\ln(1-x) = \sum_{n\geq 1} \dfrac{x^n}{n}$ for $|x|$.
b. Replace x by $-x$ and, admitting that it holds at $x = 1$, deduce $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.
c. Using the alternating series bound, how many terms of this series guarantee an error below 0.001?
d. Put $x = \frac{1}{2}$ in part (a) to get $\ln 2 = \sum_{n\geq 1} \dfrac{1}{n2^n}$. Show that 10 terms give an error below 10^{-4}.
e. Which series would you program, and why?

Question 17

Question 17 – Waiting for a six *(Section C · Problem solving)*

A fair die is rolled until the first six. The number of rolls X has $P(X = n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$, so $E[X] = \sum_{n\geq 1} n\,\frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$.

n	1	2	3	4
$P(X = n)$	0.1667	0.1389	0.1157	?

a. Complete the table and check that the probabilities add up to 1.
b. Use the ratio test to show that the series for $E[X]$ converges.
c. Differentiate $\sum x^n = \frac{1}{1-x}$ to show $\sum n x^{n-1} = \dfrac{1}{(1-x)^2}$ for $|x|$.
d. Deduce $E[X]$.
e. Find $P(X > 12)$ and interpret.

Question 18

Question 18 – Rearranging a conditionally convergent series *(Section C · Problem solving)*

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\ln 2 \approx 0.6931$. Now rearrange the same terms: one positive term followed by two negative terms.

Group k	**1**	**2**	**3**
Terms	$1 - \frac{1}{2} - \frac{1}{4}$	$\frac{1}{3} - \frac{1}{6} - \frac{1}{8}$	$\frac{1}{5} - \frac{1}{10} - \frac{1}{12}$

a. Show that the kth group equals $\dfrac{1}{2(2k-1)} - \dfrac{1}{4k}$.
b. Deduce that the rearranged series has sum $\frac{1}{2}\ln 2$.
c. Explain why this does not contradict the original sum.
d. What does Riemann's rearrangement theorem say, and why can it not happen for $\sum \frac{(-1)^n}{n^2}$?

Question 19

Question 19 – Integrating a series term by term *(Section C · Problem solving)*

The integral $I = \displaystyle\int_0^1 e^{-x^2}\,dx$ appears in probability, but e^{-x^2} has no elementary antiderivative. Engineers use a series instead.

a. Write the Maclaurin series of e^{-x^2} and state its radius of convergence.
b. Integrate term by term to show $I = \displaystyle\sum_{n\geq 0} \frac{(-1)^n}{n!(2n+1)}$.
c. Compute the sum of the first five terms.
d. Use the alternating series bound to estimate the error.
e. The true value is 0.746824. Compare.

Question 20

Question 20 – Challenge: e is irrational *(Section D · Challenge)*

We know $e = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}$. Suppose, for a contradiction, that $e = \dfrac{a}{b}$ with positive integers a, b.

a. Show that for any n, 0 by comparing the tail with a geometric series.
b. Let $x = b!\left(e - \displaystyle\sum_{k=0}^{b} \frac{1}{k!}\right)$. Show that x is an integer.
c. Show that 0.
d. Conclude.