# Answers - Sequences and series convergence

Year: University Year 1

## Answer to Question 1

a. For a rational expression, divide top and bottom by the highest power $n^2$: $a_n = \dfrac{5 - 3/n}{2 + 7/n^2}$. Since $1/n \to 0$ and $1/n^2 \to 0$, the limit is $\dfrac{5}{2}$.

b. Both terms tend to infinity, so multiply by the conjugate: $b_n = \dfrac{(n^2 + n) - n^2}{\sqrt{n^2 + n} + n} = \dfrac{n}{\sqrt{n^2 + n} + n} = \dfrac{1}{\sqrt{1 + 1/n} + 1}$. The denominator tends to $2$, so the limit is $\dfrac{1}{2}$.

c. Use the standard limit $\left(1 + \dfrac{x}{n}\right)^n \to e^x$ with $x = 2$: the limit is $e^2$.

d. Take logarithms: $\ln d_n = \dfrac{\ln n}{n} \to 0$, because $\ln n$ grows more slowly than $n$. Since the exponential is continuous, $d_n = e^{\ln d_n} \to e^0 = \mathbf{1}$.

## Answer to Question 2

a. Put everything over the common denominator: $a_n - 3 = \dfrac{3n + 1 - 3(n + 2)}{n + 2} = \dfrac{-5}{n + 2}$. Taking absolute values, $|a_n - 3| = \dfrac{5}{n + 2}$.

b. We need $\dfrac{5}{n + 2} < 0.01$, that is $n + 2 > 500$, or $n > 498$. So the smallest such value is $N = 498$: every term from $a_{499}$ onwards is within $0.01$ of 3.

c. Let $\varepsilon > 0$ be given and choose an integer $N \geq \dfrac{5}{\varepsilon}$. For every $n > N$: $|a_n - 3| = \dfrac{5}{n + 2} < \dfrac{5}{n} < \dfrac{5}{N} \leq \varepsilon$. Since such an $N$ exists for every $\varepsilon > 0$, this is exactly the definition of $a_n \to 3$.

## Answer to Question 3

a. This is a geometric series with first term $a = 5$ and ratio $r = \frac{2}{3}$. Since $|r| < 1$, it converges to $\dfrac{a}{1 - r} = \dfrac{5}{1 - 2/3} = \mathbf{15}$.

b. Use partial fractions: $\dfrac{1}{n(n + 2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n + 2}\right)$. In the partial sum most terms

cancel (it telescopes), leaving $\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right)$. As $N\to\infty$ this tends to $\frac{3}{4}$.

c. Geometric with first term $0.9$ (the sum starts at $n=1$) and ratio $0.9$: $\frac{0.9}{1-0.9}=$ **9**.

d. , a geometric series with $a=0.27$ and $r=0.01$. Its sum is $\frac{0.27}{1-0.01}=\frac{27}{99}=\frac{3}{11}$.

## Answer to Question 4

a. Since $n^2+3n>n^2$, we have $0<\frac{1}{n^2+3n}<\frac{1}{n^2}$. The series $\sum\frac{1}{n^2}$ converges ( $p=2>1$), so the given series **converges** by the comparison test.

b. Compare with $\frac{1}{\sqrt{n}}$: $\frac{1/(\sqrt{n}+1)}{1/\sqrt{n}}=\frac{\sqrt{n}}{\sqrt{n}+1}\to 1$, a finite non-zero limit. Since $\sum\frac{1}{\sqrt{n}}$ diverges ($p=\frac{1}{2}\le 1$), the series **diverges** by limit comparison.

c. For large $n$ the term behaves like $\frac{n}{n^3}=\frac{1}{n^2}$. Indeed $\frac{(n+1)/(n^3+2)}{1/n^2}=\frac{n^3+n^2}{n^3+2}\to 1$, and $\sum\frac{1}{n^2}$ converges, so the series **converges**.

d. For $n\ge 3$, $\ln n>1$, so $\frac{\ln n}{n}>\frac{1}{n}$. The harmonic series diverges, so this series **diverges** by comparison.

## Answer to Question 5

a. Factorials suggest the ratio test: $\frac{a_{n+1}}{a_n}=\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}=\frac{3}{n+1}\to 0<1$, so the series **converges** (its sum is $e^3$).

b. $\frac{a_{n+1}}{a_n}=\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}=\frac{1}{2}\left(\frac{n+1}{n}\right)^2\to\frac{1}{2}<1$: the series **converges**; the exponential beats the power.

c. $\frac{a_{n+1}}{a_n}=\frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!}=\frac{n+1}{2}\to\infty$: the series **diverges**; in fact the terms do not even tend to 0.

d. The term is an $n$th power, so use the root test: $\sqrt[n]{a_n}=\frac{2n}{3n+1}\to\frac{2}{3}<1$. The series **converges**.

## Answer to Question 6

a. $\frac{1}{\sqrt{n}}$ decreases to 0, so the alternating series test gives convergence. But $\sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$), so the convergence is **conditional**.

b. $\dfrac{n}{n^2+1}$ decreases to 0 for $n \geq 1$, so the series converges. $\dfrac{n}{n^2+1} \sim \dfrac{1}{n}$, so the absolute series diverges by limit comparison: **conditionally convergent**.

c. Ratio: [redacted], so $R = 3$. At $x = \pm 3$ the terms are $n(\pm 1)^n$, which do not tend to 0. The interval is $(-3, 3)$.

## Answer to Question 7

a. $1.125 \times 0.75 = \mathbf{0.84375\ m}$.

b. $2 + 2(1.5 + 1.125 + \ldots) = 2 + 2 \times \dfrac{1.5}{1 - 0.75} = 2 + 12 = \mathbf{14\ m}$.

c. Fall time from $h_k = 2(0.75)^k$ is $\sqrt{2h_k/g} = t_0(\sqrt{0.75})^k$ with $t_0 = \sqrt{4/9.8} \approx 0.639$ s.

d. Each bounce has a rise and a fall of equal time: total $= t_0 + 2t_0 \dfrac{r}{1-r}$ with $r = \sqrt{0.75} \approx 0.866$, giving $0.639(1 + 12.93) \approx \mathbf{8.9\ s}$.

e. The terms shrink geometrically, so their sum converges, like Zeno's paradox. In reality the ball stops after a finite number of visible bounces.

## Answer to Question 8

a. $a_2 = 0.3(100) + 100 = 130$, $a_3 = 0.3(130) + 100 = 139$.

b. By induction, $a_{n+1} = 0.3a_n + 100$ adds one more power of 0.3. Geometric sum: $a_n = \dfrac{100(1 - 0.3^n)}{0.7}$.

c. $a_5 = \dfrac{100(1 - 0.00243)}{0.7} \approx \mathbf{142.5\ mg}$; limit $\dfrac{100}{0.7} \approx \mathbf{142.9\ mg}$.

d. Yes: the level never exceeds 142.9 mg. For a limit of 120 mg we need $\dfrac{D}{0.7} = 120$, so $D = \mathbf{84\ mg}$.

## Answer to Question 9

a. The first payment (end of month 1) earns interest at the end of months 2 and 3, so it grows to $200 \times 1.004^2$. The second payment earns interest once: $200 \times 1.004$. The third has just been paid: $200$. Adding, the balance is $200(1.004^2 + 1.004 + 1)$.

b. In the same way, $B_n = 200(1 + 1.004 + \cdots + 1.004^{n-1})$, a geometric series with $n$

terms, first term 1 and ratio $1.004$. Its sum is $\dfrac{1.004^n-1}{1.004-1}$, so $B_n = 200\dfrac{1.004^n-1}{0.004}$.

c. $B_{24} = 200 \times \dfrac{1.004^{24}-1}{0.004} \approx$ **£5 027.42**. She paid in $24 \times 200 =$£4 800, so the interest earned is **£227.42**.

d. We need $B_n > 10\,000$, that is $1.004^n > 1 + \dfrac{10000 \times 0.004}{200} = 1.2$. Taking logarithms, $n > \dfrac{\ln 1.2}{\ln 1.004} \approx 45.7$, so it takes **46 months**.

## Answer to Question 10

a. In stage 1 Achilles runs the 100 m gap in 10 s; meanwhile the tortoise moves 10 m. In stage 2 he covers those 10 m in 1 s, while the tortoise moves 1 m, and so on. Each gap, and each time, is 10 times smaller than the previous one, so stage 4 takes **0.01 s**.

b. The times form a geometric series with $a = 10$ and $r = 0.1$: $10 + 1 + 0.1 + \cdots = \dfrac{10}{1-0.1} = \dfrac{100}{9} \approx$ **11.1 s**.

c. Achilles catches up when the distances are equal: $10t = 100 + t$, so $9t = 100$ and $t = \dfrac{100}{9}$. This matches part (b): an infinite number of stages can take a finite time, which resolves the paradox.

d. Distance $=$ speed $\times$ time $= 10 \times \dfrac{100}{9} = \dfrac{1000}{9} \approx$ **111.1 m**.

## Answer to Question 11

a. The number of sides is multiplied by 4 and the length divided by 3, so the perimeter is multiplied by $\frac{4}{3} > 1$: it tends to infinity.

b. $\displaystyle\sum_{n\ge1} 3 \cdot 4^{n-1}\frac{A_0}{9^n} = \frac{A_0}{3}\sum_{n\ge1}\left(\tfrac{4}{9}\right)^{n-1}$.

c. $\dfrac{A_0}{3} \cdot \dfrac{1}{1-4/9} = \dfrac{A_0}{3} \cdot \dfrac{9}{5} = \dfrac{3}{5}A_0$. Total area $A_0 + \dfrac{3}{5}A_0 = \dfrac{8}{5}A_0$.

d. The boundary becomes infinitely wiggly, yet the shape stays inside a fixed circle, so its area is bounded: a first example of a fractal.

## Answer to Question 12

a. Add the next term of the harmonic series: $H_4 = H_3 + \frac{1}{4} = 1.833 + 0.25 =$ **2.083**.

b. The top book is completely beyond the edge when the overhang exceeds one

book length: $\frac{1}{2}H_n > 1$, so $H_n > 2$. From the table, $H_3 < 2 < H_4$, so **4 books** are needed.

c. The overhang is $\frac{1}{2}H_n$, and the harmonic series diverges: $H_n$ exceeds any bound when $n$ is large enough. So any overhang, however large, can be reached in theory, although the number of books grows enormously.

d. An overhang greater than 2 needs $H_n > 4$. Since $H_{30} \approx 3.995 < 4 < H_{31} \approx 4.027$, we need **31 books**. Because $H_n$ grows like $\ln n$, each extra book length of overhang needs about $e^2 \approx 7.4$ times as many books.

## Answer to Question 13

a. $S_6 = 1.6487196$ (true value $1.6487213$).

b. The tail is $\frac{0.5^{N+1}}{(N+1)!}\left(1 + \frac{0.5}{N+2} + \dots\right)$ and each ratio is less than $0.5$, so it is bounded by a geometric series.

c. $N = 6$: bound $\frac{0.5^7}{7!} \times 2 \approx 3.1 \times 10^{-6}$, too big. $N = 7$: $\frac{0.5^8}{8!} \times 2 \approx 1.9 \times 10^{-7}$. So $N = 7$.

d. The ratio test gives $\frac{|x|}{k+1} \to 0$ for every $x$: the radius of convergence is infinite.

## Answer to Question 14

a. Apply the recurrence: $a_1 = 0.8(500) + 400 = 800$, $a_2 = 0.8(800) + 400 = 1040$ and $a_3 = 0.8(1040) + 400 = 1232$.

b. If $a_n \to L$, then also $a_{n+1} \to L$. Taking limits in the recurrence, $L = 0.8L + 400$, so $0.2L = 400$ and $L = 2000$.

c. Let $b_n = a_n - 2000$. Then $b_{n+1} = 0.8a_n + 400 - 2000 = 0.8(a_n - 2000) = 0.8b_n$, so $(b_n)$ is geometric with $b_0 = -1500$. Hence $b_n = -1500(0.8)^n$ and $a_n = 2000 - 1500(0.8)^n$.

d. Since $1500(0.8)^n > 0$, $a_n < 2000$ for all $n$. Also $a_{n+1} - a_n = 1500(0.8)^n(1 - 0.8) > 0$, so the sequence is increasing. Being increasing and bounded above, it converges by the monotone convergence theorem.

e. We need $1500(0.8)^n < 100$, that is $0.8^n < \frac{1}{15}$, so $n > \frac{\ln 15}{\ln 1.25} \approx 12.1$. The population first exceeds 1900 after **13 years**.

## Answer to Question 15

a. Because $\frac{1}{x^2}$ is decreasing, $\frac{1}{n^2}$ lies between $\int_n^{n+1}$ and $\int_{n-1}^{n}$ of $\frac{1}{x^2}$. Summing for

$n > N$ gives the inequalities.

b. $\int_a^\infty x^{-2}dx = \frac{1}{a}$.

c. $1.549768 + \frac{1}{11} =$ **1.640677** $1.549768 + 0.1 =$ **1.649768**.

d. $\pi^2/6 \approx 1.644934$ lies in the interval of width $0.009$. The error of $S_N$ alone is about $\frac{1}{N}$, so about **100 terms** give the same accuracy; the integral bounds save a lot of work.

## Answer to Question 16

a. $\displaystyle\int_0^x \frac{dt}{1-t} = -\ln(1-x)$ and $\int_0^x t^n\,dt = \frac{x^{n+1}}{n+1}$.

b. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$; at $x = 1$ we get $\ln 2$ (Abel's theorem justifies the endpoint).

c. The error is at most the first omitted term $\frac{1}{N+1}$, so we need $N \geq$ **1000 terms**.

d. The tail after 10 terms is $\displaystyle\sum_{n\geq 11} \frac{1}{n2^n}$. Indeed $S_{10} = 0.693065$ and $\ln 2 = 0.693147$.

e. The second: it converges geometrically, so 10 terms beat 1000 terms of the first.

## Answer to Question 17

a. $P(X = 4) = \frac{1}{6}\left(\frac{5}{6}\right)^3 \approx$ **0.0965**. The probabilities form a geometric series with $a = \frac{1}{6}$ and $r = \frac{5}{6}$, so their sum is $\dfrac{1/6}{1 - 5/6} = 1$, as it must be.

b. With $a_n = n \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}$, $\dfrac{a_{n+1}}{a_n} = \dfrac{n+1}{n} \cdot \dfrac{5}{6} \to \dfrac{5}{6} < 1$, so the series for $E[X]$ converges by the ratio test.

c. The geometric series $\sum x^n = \frac{1}{1-x}$ has radius of convergence 1. Inside it, a power series may be differentiated term by term, so $\displaystyle\sum n x^{n-1} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$ for $|x| < 1$.

d. With $x = \frac{5}{6}$: $\displaystyle E[X] = \frac{1}{6} \cdot \frac{1}{(1/6)^2} =$ **6 rolls**, which matches intuition.

e. $X > 12$ means the first 12 rolls are not sixes: $P(X > 12) = \left(\frac{5}{6}\right)^{12} \approx$ **0.112**. About 1 time in 9 you wait more than twice the average.

## Answer to Question 18

a. $\displaystyle\frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k} = \frac{1}{2(2k-1)} - \frac{1}{4k}$.

b. Group $k$ is $\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$, so the sum is half the alternating harmonic series: $\frac{1}{2}\ln 2 \approx 0.3466$.
c. The series is only conditionally convergent: the positive and negative parts both diverge, so the order of the terms matters.
d. A conditionally convergent series can be rearranged to have any sum, or to diverge. For an absolutely convergent series like $\sum \frac{(-1)^n}{n^2}$, every rearrangement has the same sum.

## Answer to Question 19

a. Substitute $u = -x^2$ in $e^u = \sum \frac{u^n}{n!}$: $e^{-x^2} = \sum \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$. The exponential series converges for every $u$, so $R = \infty$.
b. Inside the radius of convergence we may integrate term by term. Each term is $\frac{(-1)^n}{n!}x^{2n}$, and $\int_0^1 x^{2n}dx = \frac{1}{2n+1}$, so $I = \sum_{n\geq 0} \frac{(-1)^n}{n!(2n+1)}$.
c. The first five terms are $1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx$ **0.747487**.
d. The series alternates and its terms decrease to 0, so the error is less than the first omitted term: $\frac{1}{5! \times 11} = \frac{1}{1320} \approx 0.00076$.
e. Actual error $= 0.747487 - 0.746824 = 0.00066$, which is less than $0.00076$, as predicted.

## Answer to Question 20

a. The tail is $\frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right)$. Each fraction in the bracket is at most the corresponding power of $\frac{1}{n+1}$, so the tail is less than $\frac{1}{(n+1)!} \cdot \frac{1}{1-\frac{1}{n+1}} = \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n!\,n}$. It is positive because every term is positive.
b. If $e = \frac{a}{b}$, then $b!\,e = a(b-1)!$ is an integer. Also $\frac{b!}{k!}$ is an integer for every $k \leq b$. So $x$, a difference of integers, is an integer.
c. Apply (a) with $n = b$ and multiply by $b!$: $0 < x < \frac{b!}{b!\,b} = \frac{1}{b} \leq 1$.
d. There is no integer strictly between 0 and 1, so the assumption $e = \frac{a}{b}$ leads to a contradiction. Therefore $e$ **is irrational**.