Test answers - Probability and random variables

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Because A and B are independent, $P(A \cap B) = P(A)P(B) = 0.4 \times 0.35 = 0.14$ (1). By the addition rule, $P(A \cup B) = 0.4 + 0.35 - 0.14 = 0.61$ (1).
b. If A and B are independent, so are A and B^c; hence $P(A \mid B^c) = P(A) = 0.4$ (1). By De Morgan, $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - 0.61 = 0.39$ (1), which also equals $P(A^c)P(B^c) = 0.6 \times 0.65$, confirming that the complements are independent too.

In (a), a common error is to add $0.4 + 0.35 = 0.75$ without subtracting the overlap. In (b), the direct calculation $P(A \cap B^c) \div P(B^c) = 0.26 \div 0.65 = 0.4$ gives the same result.

Answer to Question 2 (4 marks)

a. Machine 2 makes the remaining 30% of the bolts. By the law of total probability, $P(D) = 0.7 \times 0.04 + 0.3 \times 0.10 = 0.028 + 0.030 = 0.058$ (2), so about 5.8% of all bolts are defective.
b. By Bayes' theorem, $P(M_2 \mid D) = \dfrac{P(M_2)P(D \mid M_2)}{P(D)} = \dfrac{0.030}{0.058} \approx 0.517$ (2).

   Although machine 2 makes fewer bolts, its higher defect rate means it produces just over half of the defective ones.

In (a), a tree diagram with two stages, machine then quality, shows the two paths to a defective bolt. In (b), the denominator must be the total $P(D)$.

Answer to Question 3 (4 marks)

a. $E[X] = 1(0.2) + 2(0.5) + 3(0.3) = 0.2 + 1 + 0.9 = 2.1$. Next $E[X^2] = 1(0.2) + 4(0.5) + 9(0.3) = 0.2 + 2 + 2.7 = 4.9$, so $\operatorname{Var}(X) = E[X^2] - (E[X])^2 = 4.9 - 4.41 = 0.49$ (2).
b. By linearity, $E[5 - 2X] = 5 - 2(2.1) = 5 - 4.2 = 0.8$ (1). Adding a constant does not change the spread and the factor is squared: $\operatorname{Var}(5 - 2X) = (-2)^2 \times 0.49 = 1.96$ (1). Note that the variance is multiplied by 4, not by -2: a variance can never be negative, and the standard deviation doubles from 0.7 to 1.4.

In (a), check that the probabilities add up to $0.2 + 0.5 + 0.3 = 1$. A common error is to write $\operatorname{Var}(X) = 4.9 - 2.1$, forgetting to square the mean.

Answer to Question 4 (4 marks)

a. With $n = 20$ and $p = 0.1$,
$P(X = 2) = \binom{20}{2}(0.1)^2(0.9)^{18} = 190 \times 0.01 \times 0.1501 \approx 0.285$ (2).

b. The Poisson approximation uses $\lambda = np = 20 \times 0.1 = 2$, so
$P(Y = 2) = \dfrac{2^2 e^{-2}}{2!} = 2e^{-2} \approx 0.271$ (1). The answers are close, but the approximation works best when n is large and p small; here $n = 20$ is small, so it is only rough (1). The relative error is about 5%, which would shrink for a larger sample with the same mean.

In (a), $0.9^{18} \approx 0.1501$ must be kept to enough significant figures.

Answer to Question 5 (4 marks)

a. Standardise both limits: $\dfrac{29.5 - 30}{0.4} = -1.25$ and $\dfrac{30.6 - 30}{0.4} = 1.5$ (1). Then
$P(\text{accepted}) = \Phi(1.5) - \Phi(-1.25) = 0.9332 - 0.1056 = 0.8275$ (1).

b. Assuming parts are independent, $P(\text{all 10 accepted}) = 0.8275^{10} \approx 0.151$ (1). Only about 15% of boxes are perfect, so the process needs a smaller spread, or a centre moved towards the middle of the tolerance interval (1). The interval is not symmetric about 30, so centring the process at 30.05 mm would also help.