Test answers - Number theory and discrete mathematics

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

$390 = 5 \times 72 + 30$, $72 = 2 \times 30 + 12$, $30 = 2 \times 12 + 6$, $12 = 2 \times 6$, so the gcd is 6 (2).
Backwards:
$6 = 30 - 2 \times 12 = 30 - 2(72 - 2 \times 30) = 5 \times 30 - 2 \times 72 = 5(390 - 5 \times 72) - 2 \times 72 = 5 \times 390 - 27 \times 72$ (2).

Each step of the Euclidean algorithm writes the larger number as a multiple of the smaller plus a remainder, and the last non-zero remainder is the gcd. For Bezout, work backwards from $6 = 30 - 2 \times 12$, replacing each remainder by the expression from the line above. The coefficients are $x = 5$ and $y = -27$. Check: $5 \times 390 = 1950$ and $27 \times 72 = 1944$, and the difference is 6. Prime factorisation confirms it: $390 = 2 \times 3 \times 5 \times 13$ and $72 = 2^3 \times 3^2$ share $2 \times 3 = 6$.

Answer to Question 2 (4 marks)

a. $\gcd(9, 15) = 3$ divides 6: $3x \equiv 2 \pmod 5$ so $x \equiv 4 \pmod 5$ (1), i.e. $x \equiv 4, 9, 14 \pmod{15}$ (1).
b. $2^3 = 8 \equiv 1 \pmod 7$ (1). $50 = 16 \times 3 + 2$, so $2^{50} \equiv 2^2 = 4$ (1).

In (a), a linear congruence $ax \equiv b \pmod m$ has solutions only if $\gcd(a, m)$ divides b; here 3 divides 6, so there are exactly 3 solutions modulo 15. Dividing everything by 3 gives $3x \equiv 2 \pmod 5$, and the inverse of 3 modulo 5 is 2, so $x \equiv 4$. Check: $9 \times 4 = 36 = 2 \times 15 + 6$. In (b), the powers of 2 modulo 7 repeat with period 3, so $2^{50} = (2^3)^{16} \times 2^2 \equiv 1 \times 4$.

Answer to Question 3 (4 marks)

a. 10 letters with S three times, T three times, I twice: $\dfrac{10!}{3!\,3!\,2!} = 50\,400$ (2).
b. Total $\binom{13}{5} = 1287$; with no women $\binom{7}{5} = 21$; with one woman $6\binom{7}{4} = 210$ (1). Answer $1287 - 21 - 210 = 1056$ (1).

In (a), dividing by $3!\,3!\,2!$ removes the repeated arrangements of identical letters; the letters A and C appear once each. In (b), counting at least 2 women directly needs four cases, so the complement is quicker: subtract the committees with 0 or 1 woman from the total. Direct check: $\binom{6}{2}\binom{7}{3} + \binom{6}{3}\binom{7}{2} + \binom{6}{4}\binom{7}{1} + \binom{6}{5} = 525 + 420 + 105 + 6 = 1056$.

Answer to Question 4 (4 marks)

a. $\frac{18}{2} = 9$ edges (1).
b. A tree on 7 vertices has 6 edges, so no (1).
c. There are four vertices of odd degree $(1, 3, 3, 3)$ (1); an Euler trail needs 0 or 2, so no (1).

In (a), the handshaking lemma says the sum of the degrees is twice the number of edges, $1 + 2 + 2 + 3 + 3 + 3 + 4 = 18$. In (b), every tree with n vertices has exactly $n - 1$ edges, and 9 edges is too many. In (c), an Euler trail uses every edge exactly once, so it needs exactly 0 or 2 vertices of odd degree.

Answer to Question 5 (4 marks)

a. $a_2 = 16 - 4 = 12$, $a_3 = 48 - 16 = 32$ (1).
b. $r^2 - 4r + 4 = (r - 2)^2$: a repeated root 2, so $a_n = (A + Bn)2^n$ (1). $A = 1$ and $2(1 + B) = 4$ so $B = 1$: $a_n = (n + 1)2^n$ (1). $a_3 = 4 \times 8 = 32$ (1).

In (a), use the recurrence: $a_2 = 4a_1 - 4a_0 = 16 - 4$ and $a_3 = 4a_2 - 4a_1 = 48 - 16$. In (b), the characteristic equation comes from trying $a_n = r^n$. Because 2 is a repeated root, the second solution is $n2^n$, just as for a differential equation with a repeated root. The initial conditions give $A = a_0 = 1$ and $2(A + B) = a_1 = 4$. The formula also gives $a_2 = 3 \times 4 = 12$, matching part (a).