Answers - Number theory and discrete mathematics

Year: University Year 1

Answer to Question 1

a. $1071 = 2 \times 462 + 147$, then $462 = 3 \times 147 + 21$, then $147 = 7 \times 21 + 0$. The last non-zero remainder is $\gcd = \mathbf{21}$.
b. $21 = 462 - 3 \times 147 = 462 - 3(1071 - 2 \times 462) = 7 \times 462 - 3 \times 1071$. So $x = -3$, $y = 7$: indeed $7 \times 462 - 3 \times 1071 = 3234 - 3213 = 21$.
c. $\text{lcm} = \dfrac{1071 \times 462}{21} = \dfrac{494\,802}{21} = \mathbf{23\ 562}$.

Answer to Question 2

a. Euclid: $26 = 2 \times 11 + 4$, $11 = 2 \times 4 + 3$, $4 = 3 + 1$. Backwards: $1 = 4 - 3 = 4 - (11 - 2 \times 4) = 3 \times 4 - 11 = 3(26 - 2 \times 11) - 11 = 3 \times 26 - 7 \times 11$. So $-7 \times 11 \equiv 1$ and the inverse is $-7 \equiv \mathbf{19}$. Check: $11 \times 19 = 209 = 8 \times 26 + 1$.
b. Multiply both sides by 19: $x \equiv 5 \times 19 = 95 \equiv 95 - 78 = \mathbf{17} \pmod{26}$. Check: $11 \times 17 = 187 = 7 \times 26 + 5$.
c. $\gcd(6, 10) = 2$ divides 4, so there are 2 solutions modulo 10. Divide by 2: $3x \equiv 2 \pmod{5}$. The inverse of 3 mod 5 is 2, so $x \equiv 4 \pmod{5}$. Modulo 10: $x \equiv 4$ **or** $x \equiv 9$.

Answer to Question 3

a. 13 is prime and does not divide 5, so $5^{12} \equiv 1 \pmod{13}$. Since $100 = 8 \times 12 + 4$, $5^{100} \equiv 5^4 = 625$. And $625 = 48 \times 13 + 1$, so $5^{100} \equiv \mathbf{1} \pmod{13}$.
b. $360 = 2^3 \times 3^2 \times 5$, so $\varphi(360) = 360\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 360 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = \mathbf{96}$.
c. $\gcd(7, 10) = 1$ and $\varphi(10) = 4$, so $7^4 \equiv 1 \pmod{10}$. $222 = 55 \times 4 + 2$, so $7^{222} \equiv 7^2 = 49 \equiv 9$. The last digit is **9**.

Answer to Question 4

a. The moduli are pairwise coprime, so there is one solution modulo 105. From $x \equiv 2 \pmod{3}$ and $x \equiv 2 \pmod{7}$ we get $x \equiv 2 \pmod{21}$: $x = 2, 23, 44, 65, 86$. Of these, $23 \equiv 3 \pmod{5}$. So $x \equiv 23 \pmod{105}$.
b. $x = 4 + 9k$ must satisfy $4 + 9k \equiv 1 \pmod{4}$, i.e. $k \equiv -3 \equiv 1 \pmod{4}$ (since $9 \equiv 1$). So $k = 1$ and $x = 13$. Answer: $x \equiv 13 \pmod{36}$. Check: $13 = 3 \times 4 + 1$ and

$13 = 9 + 4$.

Answer to Question 5

a. 6 letters with A repeated 3 times and N twice: $\frac{6!}{3!\,2!} = \frac{720}{12} = \mathbf{60}$.
b. Order does not matter: $\binom{12}{4} = \frac{12 \times 11 \times 10 \times 9}{24} = \mathbf{495}$.
c. Inclusion-exclusion with $\lfloor 1000/3 \rfloor = 333$, $\lfloor 1000/5 \rfloor = 200$, $\lfloor 1000/7 \rfloor = 142$, $\lfloor 1000/15 \rfloor = 66$, $\lfloor 1000/21 \rfloor = 47$, $\lfloor 1000/35 \rfloor = 28$, $\lfloor 1000/105 \rfloor = 9$:
$333 + 200 + 142 - 66 - 47 - 28 + 9 = \mathbf{543}$.

Answer to Question 6

a. By the handshaking lemma, the number of edges is $\frac{4+4+3+3+2+2}{2} = \mathbf{9}$.
There are exactly two vertices of odd degree, so there is an **Euler trail** (starting and ending at the odd vertices) but no Euler circuit.
b. A tree with n vertices has $n - 1$ edges: **14**.
c. Every vertex of K_n has degree $n - 1$, which must be even: n **odd** (and $n \geq 3$). K_6 has $\binom{6}{2} = \mathbf{15}$ edges.

Answer to Question 7

a. A whole number of tiles must fit along each side, so the side length divides both 336 cm and 252 cm.
b. $336 = 1 \times 252 + 84$ and $252 = 3 \times 84$. So $\gcd(336, 252) = \mathbf{84\ cm}$.
c. $\frac{336}{84} \times \frac{252}{84} = 4 \times 3 = \mathbf{12\ tiles}$.
d. $12 \times 7.50 = \mathbf{£90}$. With 28 cm tiles: $12 \times 9 = 108$ tiles costing $108 \times 1.20 = \mathbf{£129.60}$. The large tiles are cheaper.

Answer to Question 8

a. $\gcd(5, 8) = 1$ divides 79, so by Bezout there are integer solutions.
b. Multiply by 79: $x_0 = -237$, $y_0 = 158$ is a solution.
c. General solution: $x = -237 + 8t$, $y = 158 - 5t$. We need $t \geq 30$ (so $x \geq 3$) and $t \leq 31$ (so $y \geq 3$). $t = 30$ gives $(3, 8)$ and $t = 31$ gives $(11, 3)$. So **3 five-pence and 8 eight-pence stamps, or 11 five-pence and 3 eight-pence stamps**.
d. For 27: $y = 0, 1, 2, 3$ gives $27, 19, 11, 3$ remaining, none a multiple of 5. From 28 to 32 we have $28 = 4 \times 5 + 8$, $29 = 5 + 3 \times 8$, $30 = 6 \times 5$, $31 = 3 \times 5 + 2 \times 8$,

$32 = 4 \times 8$. Five consecutive amounts work, and adding 5p stamps reaches every larger amount. $5 \times 8 - 5 - 8 = 27$.

Answer to Question 9

a. $365 = 52 \times 7 + 1$, so the day moves on by 1: **Saturday**.
b. $100 = 14 \times 7 + 2$, so two days after Friday: **Sunday** (11 April 2027).
c. $366 \equiv 2 \pmod{7}$, so two days after Saturday: **Monday**.
d. Each ordinary year shifts the weekday by 1 and each leap year by 2. A calendar can be reused only when the total shift is a multiple of 7 and the year has the same leap status; this usually takes 6 or 11 years.

Answer to Question 10

a. $0 + 27 + 0 + 42 + 24 + 0 + 24 + 3 + 10 = \mathbf{130}$.
b. We need $130 + d_{10} \equiv 0 \pmod{11}$. $130 = 11 \times 11 + 9$, so $d_{10} = \mathbf{2}$. The ISBN is 0-306-40615-2.
c. The sum changes by $6 \times (7 - 4) = 18$, and $18 \not\equiv 0 \pmod{11}$, so the total is no longer a multiple of 11.
d. A wrong digit in position with weight w changes the sum by $w \times e$ where $1 \le w \le 10$ and $1 \le |e| \le 10$. Since 11 is prime and divides neither w nor e, it does not divide we.

Answer to Question 11

a. $\varphi(55) = (5 - 1)(11 - 1) = \mathbf{40}$.
b. $3d \equiv 1 \pmod{40}$: $3 \times 27 = 81 = 2 \times 40 + 1$, so $d = \mathbf{27}$.
c. $8^3 = 512 = 9 \times 55 + 17$, so $c = \mathbf{17}$.
d. $c^d = m^{3 \times 27} = m^{81} = m^{2 \times 40 + 1} \equiv m$ by Euler's theorem (as $\gcd(8, 55) = 1$), so $17^{27} \equiv 8 \pmod{55}$. Anyone who can factorise n can find d; with large primes factorising is far too slow, which keeps the key secret.

Answer to Question 12

a. $36^8 = \mathbf{2\,821\,109\,907\,456}$ (about 2.8×10^{12}).
b. Complement: subtract the passwords with letters only.
$36^8 - 26^8 = 2\,821\,109\,907\,456 - 208\,827\,064\,576 = \mathbf{2\,612\,282\,842\,880}$.
c. $\dfrac{2.82 \times 10^{12}}{10^9} \approx 2821$ s, about **47 minutes**: far too weak.
d. $62^8 \approx 2.18 \times 10^{14}$, so $\left(\dfrac{62}{36}\right)^8 \approx$ **77 times** more.

Answer to Question 13

a. The complete graph K_8, with $\binom{8}{2} = 28$ **matches**.

b. Each vertex has degree **7**: every team plays 7 matches. Check: $\frac{8 \times 7}{2} = 28$.

c. Each team needs 7 evenings for its 7 matches. With 8 teams, 4 matches can be played each evening, and $28 \div 4 = 7$ evenings is enough with a good timetable.

d. $\binom{9}{2} = 36$ **matches**. With 9 teams (an odd number) one team must rest each evening, so at most 4 matches are played per evening and 9 evenings are needed.

Answer to Question 14

a. Handshaking lemma: $\frac{2+3+4+2+3+2}{2} = 8$ **streets**.

b. No: an Euler circuit needs every degree even, but B and E have odd degree.

c. Exactly two vertices have odd degree, so an Euler trail exists: it must **start at** B **and finish at** E (or the reverse).

d. A new street **between** B **and** E makes both degrees even (4), and then an Euler circuit exists, so the lorry can start and end at A.

Answer to Question 15

a. The left end is covered either by one domino across (leaving a $2 \times (n-1)$ strip: t_{n-1} ways) or by two dominoes along, one above the other (leaving $2 \times (n-2)$: t_{n-2} ways).

b. $t_6 = 13$, $t_7 = 21$, $t_8 = 34$, $t_9 = 55$, $t_{10} = \mathbf{89}$.

c. $r^2 - r - 1 = 0$ gives $r = \frac{1 \pm \sqrt{1+4}}{2}$.

d. True for $n = 1, 2$ ($1 \leq 1$, $2 \leq 2$). If true for $n-1$ and $n-2$, then $t_n \leq 2^{n-2} + 2^{n-3} = 3 \times 2^{n-3}$. By strong induction the bound holds for all n.

e. The ratio of consecutive terms tends to $\phi \approx$ **1.618**, since $t_n \approx \frac{\phi^{n+1}}{\sqrt{5}}$ and $|\psi|$ makes the other term negligible.

Answer to Question 16

a. $a_2 = 8 + 6 = 14$, $a_3 = 14 + 48 = 62$, $a_4 = 62 + 84 = \mathbf{146}$.

b. $r^2 = r + 6$, i.e. $(r-3)(r+2) = 0$: $r = 3$ **and** $r = -2$.

c. $A + B = 1$ and $3A - 2B = 8$. Adding twice the first: $5A = 10$, so $A = 2$ and

$B = -1$: $a_n = 2 \times 3^n - (-2)^n$.

d. $a_4 = 2 \times 81 - 16 = 146$, as in the table. $a_{10} = 2 \times 59\,049 - 1024 =$ **117 074**.

e. The 3^n term dominates: the population roughly triples every season, with the $(-2)^n$ term causing a small alternating correction.

Answer to Question 17

a. $x \equiv 3 \pmod 4$, $x \equiv 2 \pmod 5$, $x \equiv 1 \pmod 7$.

b. The moduli 4, 5, 7 are pairwise coprime and $4 \times 5 \times 7 = 140$.

c. From $x \equiv 1 \pmod 7$: $x = 1, 8, 15, 22, 29, \ldots$. Those $\equiv 2 \pmod 5$: $22, 57, 92, 127, \ldots$ (step 35). Of $22, 57, 92, 127$, only $127 \equiv 3 \pmod 4$. So $x \equiv 127 \pmod{140}$.

d. Between 100 and 300: **127 or 267**.

e. **267 members**. Check: $267 = 66 \times 4 + 3 = 53 \times 5 + 2 = 38 \times 7 + 1$.

Answer to Question 18

a. The degrees are $3, 3, 3, 2, 3, 2$, with sum 16, so there are **8 edges**: MP, MC, MH, PC, PE, CB, BE, EH.

b. M, P and C all clash with each other (a triangle), so they need three different sessions.

c. Session 1: **M, B**. Session 2: **P, H**. Session 3: **C, E**. No pair inside a session clashes.

d. Three sessions are necessary and sufficient: the chromatic number is **3**.

e. Adding the edge BH: the colouring above puts B in 1 and H in 2, so it still works. **Yes**, 3 sessions are enough.

Answer to Question 19

a. $6, 48, 342, 2400$ are multiples of 6 ($1, 8, 57, 400$ times) and $8, 80, 728, 6560$ are multiples of 8 ($1, 10, 91, 820$ times).

b. Base: $7 - 1 = 6$. Step: if $7^k - 1 = 6m$, then $7^{k+1} - 1 = 7(7^k - 1) + 6 = 42m + 6 = 6(7m + 1)$.

c. Base: $9 - 1 = 8$. Step: if $3^{2k} - 1 = 8m$, then $3^{2k+2} - 1 = 9(3^{2k} - 1) + 8 = 8(9m + 1)$.

d. Base: $1 \cdot 1! = 1 = 2! - 1$. Step: if the sum up to k is $(k+1)! - 1$, adding $(k+1)(k+1)!$ gives $(k+1)!(1 + k + 1) - 1 = (k+2)! - 1$.

Answer to Question 20

a. $25 + 12 + 6 + 3 + 1 =$ **47**, so 2^{47} divides $50!$.

b. Zeros come from factors $10 = 2 \times 5$, and fives are rarer. For $100!$: $20 + 4 =$ **24** zeros. For $1000!$: $200 + 40 + 8 + 1 =$ **249** zeros.

c. $6! = 720 = 102 \times 7 + 6 \equiv -1 \pmod{7}$.
$10! = 3\,628\,800 = 329\,890 \times 11 + 10 \equiv -1 \pmod{11}$.

d. Both a and b are distinct factors of $(n-1)!$, so $ab = n$ divides it and $(n-1)! \equiv 0 \not\equiv -1$. (If $n = p^2$ with $p > 2$, then p and $2p$ both appear, giving the same conclusion; $n = 4$ gives $3! \equiv 2$.)