Test answers - Multivariable calculus

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. $f_x = 2xe^y$, $f_y = x^2e^y$, $f_{xy} = 2xe^y$ (2).
b. At $(2, 0)$: $f_x = 4$, $f_y = 4$ (1). The plane is $z - 4 = 4(x - 2) + 4y$, i.e. $z = 4x + 4y - 4$ (1).

In (a), to find f_x treat y as a constant, so e^y is a constant factor; to find f_y treat x as a constant, so x^2 is a constant factor. Differentiating f_x with respect to y gives f_{xy}, and $f_{yx} = \partial_x(x^2e^y) = 2xe^y$ is the same, as Clairaut's theorem predicts. In (b), the tangent plane is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$; check that $(2, 0, 4)$ lies on it: $8 + 0 - 4 = 4$.

Answer to Question 2 (4 marks)

$f_x = 3x^2 - 12 = 0$ gives $x = \pm 2$; $f_y = 2y - 4 = 0$ gives $y = 2$ (2). $f_{xx} = 6x$, $f_{yy} = 2$, $f_{xy} = 0$, so $D = 12x$ (1). At $(2, 2)$: $D = 24 > 0$, $f_{xx} > 0$: minimum. At $(-2, 2)$: D: saddle (1).

The critical points are where both first partial derivatives vanish, so solve the two equations together, giving $(2, 2)$ and $(-2, 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 6x \times 2 - 0 = 12x$. At $(2, 2)$, $D > 0$ and $f_{xx} = 12 > 0$, so it is a local minimum with value $8 - 24 + 4 - 8 = -20$. At $(-2, 2)$, $D = -24 < 0$, so it is a saddle point: the function increases in some directions and decreases in others.

Answer to Question 3 (4 marks)

$(y^2, 2xy) = \lambda(1, 1)$ so $y^2 = 2xy$ (1), giving $y = 0$ or $y = 2x$ (1). With $y = 2x$: $3x = 3$, $x = 1$, $y = 2$ (1). Maximum value $f = 1 \times 4 = 4$; the other candidate $y = 0$ gives $f = 0$ (1).

The method sets $\nabla f = \lambda \nabla g$ with constraint $g(x, y) = x + y = 3$, whose gradient is $(1, 1)$. Equating the two components gives $y^2 = \lambda$ and $2xy = \lambda$. The endpoints of the constraint segment, $(3, 0)$ and $(0, 3)$, both give $f = 0$, so the maximum on the closed segment is 4. Check by substitution: $f = (3 - y)y^2$ has derivative $6y - 3y^2 = 0$ at $y = 2$.

Answer to Question 4 (4 marks)

a. Inner: $\int_0^1 (x + 2y)dx = \frac{1}{2} + 2y$ (1). Outer: $\int_0^2 (\frac{1}{2} + 2y)dy = 1 + 4 = 5$ (1).
b. $\int_0^{2\pi}\int_0^1 \frac{r\,dr}{1 + r^2}\,d\theta = 2\pi \cdot \frac{1}{2}\ln 2$ (1) $= \pi \ln 2$ (1).

In (a), integrate first with respect to x, treating y as a constant: $\left[\frac{x^2}{2} + 2xy\right]_0^1 = \frac{1}{2} + 2y$. Then integrate with respect to y: $\left[\frac{y}{2} + y^2\right]_0^2 = 1 + 4 = 5$. In (b), change to polar coordinates with $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$; a common error is to forget the Jacobian factor r. The disc corresponds to $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Substituting $u = 1 + r^2$ gives $\int_0^1 \frac{r}{1+r^2}dr = \frac{1}{2}\ln 2$.

Answer to Question 5 (4 marks)

a. $V = 24$ m^3; $V_x = yz = 6$, $V_y = xz = 8$, $V_z = xy = 12$ (2).
b. $|dV| \le (6 + 8 + 12)(0.05) = 1.3$ m^3 (2), about 5.4% of V.

In (a), each partial derivative is the product of the other two sides, because V is linear in each variable. In (b), the total differential is $dV = V_x\,dx + V_y\,dy + V_z\,dz$, and the worst case takes every error with the same sign, $|dx| = |dy| = |dz| = 0.05$. Check against the exact worst case: $4.05 \times 3.05 \times 2.05 \approx 25.32$, which is about 1.32 m³ more than 24, close to the estimate. The relative error is $1.3 \div 24 \approx 5.4\%$.