Worksheet - Multivariable calculus

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Partial derivatives *(Section A · Skills practice)*

Find the first partial derivatives of each function and, for part (b), all second partial derivatives.

a. $f(x, y) = x^2y + 3y^3$ [2 marks]
b. $g(x, y) = x^2y^3$ [4 marks]
c. $h(x, y) = \sin(xy)$: verify that $h_{xy} = h_{yx}$. [4 marks]

Question 2

Question 2 – Gradient and directional derivatives *(Section A · Skills practice)*

Answer each part.

a. Find ∇f for $f = x^2y$ and the directional derivative at $(1, 2)$ in the direction of $(3, 4)$. [4 marks]
b. For $f = xyz$, find ∇f at $(1, 2, 2)$ and the greatest rate of increase there. [3 marks]

Question 3

Question 3 – Tangent planes and the chain rule *(Section A · Skills practice)*

Answer each part.

a. Find the tangent plane to $z = x^2 + y^2$ at $(1, 1, 2)$. [3 marks]
b. If $z = x^2y$ with $x = t^2$ and $y = t^3$, find $\dfrac{dz}{dt}$ in two ways. [4 marks]
c. If $w = x^2 + y^2$ with $x = \cos t$ and $y = \sin t$, find $\dfrac{dw}{dt}$ and explain the result. [3 marks]

Question 4

Question 4 – Critical points *(Section A · Skills practice)*

Find and classify all critical points of each function.

a. $f = x^2 + y^2 - 2x + 4y$ [4 marks]
b. $g = x^2 - y^2$ [3 marks]

c. $k = x^3 - 3x + y^2$ [5 marks]

Question 5

Question 5 – Lagrange multipliers *(Section A · Skills practice)*

Use the method of Lagrange multipliers.

a. Maximise $f = xy$ subject to $x + y = 10$. [4 marks]
b. Find the maximum and minimum of $f = x + y$ on the circle $x^2 + y^2 = 1$. [5 marks]

Question 6

Question 6 – Double integrals *(Section A · Skills practice)*

Evaluate each integral.

a. $\int_0^1 \int_0^2 xy \, dy \, dx$ [2 marks]
b. $\int_0^1 \int_0^x (x + y) \, dy \, dx$ [3 marks]
c. $\int_0^1 \int_y^1 e^{x^2} \, dx \, dy$ by reversing the order [4 marks]
d. $\iint_D (x^2 + y^2) \, dA$ where D is the disc $x^2 + y^2 \leq 4$ [3 marks]

Question 7

Question 7 – Walking up a hill *(Section B · Real-life practice)*

The height of a hill in metres is $h(x, y) = 100 - x^2 - 2y^2$, where x, y are in hundreds of metres east and north of the summit-marker. A hiker is at $(3, 2)$.

a. Find the height at the hiker's position.
b. Find ∇h there. In which direction is the climb steepest?
c. What is the steepest slope there (metres of height per hundred metres horizontally)?
d. Find the rate of change of height if the hiker walks directly east.

Question 8

Question 8 – Marginal products in a factory *(Section B · Real-life practice)*

A factory's output is $P(L, K) = 10L^{0.6}K^{0.4}$, where L is the number of labour hours (per week) and K the machine hours. The current values are $L = 100$ and $K = 50$.

a. Find P at the current values.

b. Find $\frac{\partial P}{\partial L}$ and $\frac{\partial P}{\partial K}$ and interpret them (marginal products).
c. Use the linear approximation to estimate the change in output from adding 2 labour hours and 1 machine hour.
d. Show that $L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P$ (Euler's theorem for this function) and check at the current values.

Question 9

Question 9 – The cheapest open box *(Section B · Real-life practice)*

A company wants an open-topped box with a square base of side x m and height z m with a volume of 32 m^3, using as little material as possible.

a. Write the surface area $S = x^2 + 4xz$ and the constraint $x^2 z = 32$.
b. Eliminate z and minimise $S(x)$ for $x > 0$.
c. Verify using Lagrange multipliers that $\nabla S = \lambda \nabla g$ has the same solution.
d. Find the dimensions, the minimum area, and the cost at £12 per m^2.

Question 10

Question 10 – Errors in measurements *(Section B · Real-life practice)*

The radius and height of a cylindrical can are measured as $r = 5.0$ cm and $h = 12.0$ cm with possible errors ± 0.1 cm and ± 0.2 cm respectively. The volume is $V = \pi r^2 h$.

a. Find V using the measured values.
b. Find V_r and V_h at $(5, 12)$.
c. Use the total differential $dV = V_r\,dr + V_h\,dh$ to estimate the maximum error in V.
d. Express this as a percentage of V and say which measurement contributes more.

Question 11

Question 11 – The gas law *(Section B · Real-life practice)*

For one mole of an ideal gas, $PV = RT$ with $R = 8.31$ J/(mol K). Treat the pressure as a function $P(T, V) = \frac{RT}{V}$. A sample has $T = 300$ K and $V = 0.025$ m^3.

a. Find the pressure.
b. Find $\frac{\partial P}{\partial T}$ and $\frac{\partial P}{\partial V}$ and interpret them.
c. Estimate the change in pressure if the temperature rises by 2 K and the volume shrinks by 0.001 m^3.
d. Show that $\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1$.

Question 12

Question 12 – A consumer's best choice *(Section B · Real-life practice)*

A student has £60 a month for music. Downloads cost £3 each (x of them) and streaming credits cost £2 each (y of them). The student's satisfaction is modelled by $U(x, y) = xy$.

a. Write the budget constraint $g(x, y) = 3x + 2y = 60$.
b. Use Lagrange multipliers to find the best choice of x and y.
c. Find the value of λ and interpret it.
d. Check the interpretation by raising the budget to £61.

Question 13

Question 13 – Rainfall over a district *(Section B · Real-life practice)*

During a storm the depth of rain (in mm) at the point (x, y) of a rectangular district, $0 \le x \le 4$ km and $0 \le y \le 3$ km, is $r(x, y) = 2 + x + y$.

a. Find the rainfall at the corners $(0, 0)$ and $(4, 3)$.
b. Evaluate $\displaystyle\int_0^4 \int_0^3 (2 + x + y)\, dy\, dx$.
c. The integral is in mm km^2. Convert to a volume in m^3 (1 mm km^2 = 1 000 m^3).
d. Find the mean depth over the district.

Question 14

Question 14 – A fountain basin *(Section B · Real-life practice)*

The water surface of a decorative fountain forms a dome with height $z = 4 - x^2 - y^2$ metres above the plane $z = 0$, over the disc $x^2 + y^2 \le 4$ (lengths in metres).

a. Explain why polar coordinates suit this region and write the integral for the volume under the dome.
b. Evaluate the volume.
c. Find the maximum height and the radius of the base.
d. Find the mean height of the dome over its base.

Question 15

Question 15 – A landscape with a valley and a pass *(Section C · Problem solving)*

A landscape designer models the height of a garden terrain by $f(x, y) = x^3 + y^3 - 3xy$ (in metres, with x, y in tens of metres).

Point	f
$(0, 0)$	0

Point	f
$(1, 1)$	-1
$(2, 2)$	4
$(-1, -1)$	-5

a. Find f_x and f_y and solve $f_x = f_y = 0$ to find the critical points.
b. Find the second derivatives and the discriminant $D = f_{xx} f_{yy} - f_{xy}^2$.
c. Classify each critical point.
d. Explain why this surface has no global minimum, and check the table entry at $(-1, -1)$.

Question 16

Question 16 – Best position for a router *(Section C · Problem solving)*

A router must be placed on the straight corridor $3x + 4y = 25$ (lengths in metres). A server sits at the origin and the router should be as close to it as possible to keep the cable short.

a. Write the squared distance $f = x^2 + y^2$ and the constraint $g = 3x + 4y$.
b. Use Lagrange multipliers to find the best position.
c. Find the shortest cable length and compare with the formula for the distance from a point to a line.
d. The corridor is now the elliptical hall $x^2 + 4y^2 = 25$. Find the points on it farthest from the origin.

Question 17

Question 17 – Reversing the order of integration *(Section C · Problem solving)*

A heat sensor reading over the triangular region T with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$ is modelled by $I = \int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

a. Describe the region T using inequalities and sketch it in words.
b. Explain why the integral cannot be done directly in this order.
c. Reverse the order and evaluate I exactly.
d. Find the area of T and the average value of e^{x^2} over T.

Question 18

Question 18 – The Gaussian integral *(Section C · Problem solving)*

The normal distribution depends on $J = \int_{-\infty}^{\infty} e^{-x^2} dx$. The trick is to compute J^2 as a double integral.

a. Show that $J^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$.
b. Convert to polar coordinates and evaluate the double integral.
c. Deduce the value of J.
d. Hence show that $\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$, explaining why the factor $\frac{1}{\sqrt{2\pi}}$ appears in the normal density.

Question 19

Question 19 – A drone in a temperature field *(Section C · Problem solving)*

The temperature in a greenhouse is $T(x, y) = x^2 y$ (in °C, with positions in metres). A drone flies around the circle $x = 2\cos t$, $y = 2\sin t$ for $t \geq 0$ (in minutes).

a. Find T_x and T_y.
b. Use the chain rule to find $\dfrac{dT}{dt}$ in terms of t.
c. Evaluate $\dfrac{dT}{dt}$ at $t = \dfrac{\pi}{4}$ and say whether the drone is warming or cooling.
d. Check by substituting into T and differentiating directly: $T(t) = 8\cos^2 t \sin t$.

Question 20

Question 20 – Challenge: harmonic functions and waves *(Section D · Challenge)*

A function $u(x, y)$ is harmonic if $u_{xx} + u_{yy} = 0$. A function $v(x, t)$ solves the wave equation if $v_{tt} = c^2 v_{xx}$.

a. Show that $u = x^2 - y^2$ and $u = e^x \cos y$ are harmonic.
b. Show that $u = \ln(x^2 + y^2)$ is harmonic for $(x, y) \neq (0, 0)$.
c. Show that $v = f(x - ct)$ solves the wave equation for any twice-differentiable f.
d. For $f = \sin$ and $c = 2$ describe the wave $\sin(x - 2t)$: its period in x and its speed.