Answers - Multivariable calculus

Year: University Year 1

Answer to Question 1

a. Treat y as a constant for f_x: $f_x = 2xy$. Treat x as a constant for f_y: $f_y = x^2 + 9y^2$.

b. $g_x = 2xy^3$, $g_y = 3x^2y^2$. Then $g_{xx} = 2y^3$, $g_{yy} = 6x^2y$, $g_{xy} = \dfrac{\partial}{\partial y}(2xy^3) = 6xy^2$ and $g_{yx} = \dfrac{\partial}{\partial x}(3x^2y^2) = 6xy^2$.

c. $h_x = y\cos(xy)$ and $h_y = x\cos(xy)$. Then $h_{xy} = \cos(xy) - xy\sin(xy)$ and $h_{yx} = \cos(xy) - xy\sin(xy)$. They are equal, as Clairaut's theorem predicts for continuous second derivatives.

Answer to Question 2

a. $\nabla f = (2xy,\, x^2)$, which is $(4, 1)$ at $(1, 2)$. The unit vector is $\mathbf{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$. So $D_{\mathbf{u}}f = 4 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} = \dfrac{16}{5} = \mathbf{3.2}$.

b. $\nabla f = (yz,\, xz,\, xy) = (4, 2, 2)$ at $(1, 2, 2)$. The greatest rate of increase is $|\nabla f| = \sqrt{16 + 4 + 4} = \sqrt{24} \approx \mathbf{4.90}$, in the direction of the gradient.

Answer to Question 3

a. $z_x = 2x = 2$ and $z_y = 2y = 2$ at $(1, 1)$. The tangent plane is $z - 2 = 2(x - 1) + 2(y - 1)$, that is $z = 2x + 2y - 2$.

b. Chain rule:
$\dfrac{dz}{dt} = z_x\dfrac{dx}{dt} + z_y\dfrac{dy}{dt} = 2xy \cdot 2t + x^2 \cdot 3t^2 = 2t^5 \cdot 2t + t^4 \cdot 3t^2 = 4t^6 + 3t^6 = 7t^6$.
Directly: $z = t^4 \cdot t^3 = t^7$, so $\dfrac{dz}{dt} = 7t^6$. The two methods agree.

c. $\dfrac{dw}{dt} = 2x(-\sin t) + 2y\cos t = -2\cos t\sin t + 2\sin t\cos t = \mathbf{0}$. The point moves on the unit circle where $w = x^2 + y^2 = 1$ is constant.

Answer to Question 4

a. $f_x = 2x - 2 = 0$ and $f_y = 2y + 4 = 0$ give $(1, -2)$. $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$, so $D = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ with $f_{xx} > 0$: a local **minimum** with value $1 + 4 - 2 - 8 = -5$.

b. $g_x = 2x$, $g_y = -2y$ give $(0,0)$. $g_{xx} = 2$, $g_{yy} = -2$, so $D = -4$: a **saddle point**.
c. $k_x = 3x^2 - 3 = 0$ gives $x = \pm 1$ and $k_y = 2y = 0$ gives $y = 0$. Critical points $(1,0)$, $(-1,0)$. $k_{xx} = 6x$, $k_{yy} = 2$, $k_{xy} = 0$. At $(1,0)$: $D = 12 > 0$, $k_{xx} = 6 > 0$: **minimum** ($k = -2$). At $(-1,0)$: $D = -12$: **saddle** ($k = 2$).

Answer to Question 5

a. Set $\nabla f = \lambda \nabla g$ with $g = x + y$: $(y, x) = \lambda(1,1)$, so $y = x = \lambda$. The constraint gives $x = y = 5$ and the maximum is **25**.
b. $(1,1) = \lambda(2x, 2y)$ gives $x = y = \dfrac{1}{2\lambda}$. Then $2x^2 = 1$, so $x = y = \pm\dfrac{1}{\sqrt{2}}$. The maximum is $\dfrac{2}{\sqrt{2}} = \sqrt{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and the minimum is $-\sqrt{2}$ at the opposite point.

Answer to Question 6

a. Inner: $\int_0^2 xy\,dy = 2x$. Outer: $\int_0^1 2x\,dx = \mathbf{1}$.
b. Inner: $\left[xy + \dfrac{y^2}{2}\right]_0^x = x^2 + \dfrac{x^2}{2} = \dfrac{3x^2}{2}$. Outer: $\int_0^1 \dfrac{3x^2}{2}dx = \dfrac{1}{2}$.
c. The region is $0 \le y \le x \le 1$. Reversing: $\int_0^1 \int_0^x e^{x^2} dy\,dx = \int_0^1 x e^{x^2} dx = \left[\frac{1}{2}e^{x^2}\right]_0^1 = \dfrac{e-1}{2}$.
d. Polar: $\int_0^{2\pi} \int_0^2 r^2 \cdot r\,dr\,d\theta = 2\pi \cdot \dfrac{2^4}{4} = 8\pi$.

Answer to Question 7

a. $h(3,2) = 100 - 9 - 8 = \mathbf{83}$ **m**.
b. $\nabla h = (-2x, -4y) = (-6, -8)$ at $(3,2)$. The steepest ascent is in the direction $(-6,-8)$, that is towards the south-west, towards the summit at the origin.
c. $|\nabla h| = \sqrt{36 + 64} = \mathbf{10}$ metres of height per hundred metres horizontally (gradient 0.1).
d. Walking east is the direction $(1,0)$: $D = \nabla h \cdot (1,0) = -6$, so the height drops by 6 m per 100 m.

Answer to Question 8

a. $P = 10 \times 100^{0.6} \times 50^{0.4} \approx \mathbf{757.9}$ units.

b. $P_L = 6L^{-0.4}K^{0.4} \approx$ **4.55** and $P_K = 4L^{0.6}K^{-0.6} \approx$ **6.06**. One extra labour hour adds about 4.55 units of output; one extra machine hour adds about 6.06.
c. $\Delta P \approx P_L\,\Delta L + P_K\,\Delta K = 4.55 \times 2 + 6.06 \times 1 \approx$ **15.2 units**.
d. $LP_L + KP_K = 6L^{0.6}K^{0.4} + 4L^{0.6}K^{0.4} = 10L^{0.6}K^{0.4} = P$. Numerically $100(4.547) + 50(6.063) = 454.7 + 303.1 = 757.8$.

Answer to Question 9

a. The base has area x^2 and the four sides have total area $4xz$. The constraint is the volume.
b. $z = \dfrac{32}{x^2}$, so $S = x^2 + \dfrac{128}{x}$. $S' = 2x - \dfrac{128}{x^2} = 0$ gives $x^3 = 64$, so $x = 4$. $S'' = 2 + \dfrac{256}{x^3} > 0$: a minimum.
c. With $S = x^2 + 4xz$ and $g = x^2z$: $(2x + 4z,\ 4x) = \lambda(2xz,\ x^2)$ in (x, z). The second component gives $\lambda = 4/x$; the first then gives $2x + 4z = 8z$, so $x = 2z$. With $x^2z = 32$: $4z^3 = 32$, $z = 2$, $x = 4$. The same solution.
d. The base is 4×4 m and the height is 2 m. $S = 16 + 4 \times 4 \times 2 =$ **48 m**^{2}, costing $48 \times 12 =$ **£576**.

Answer to Question 10

a. $V = \pi(25)(12) = 300\pi \approx$ **942.5 cm**^{3}.
b. $V_r = 2\pi rh = 120\pi$ and $V_h = \pi r^2 = 25\pi$.
c. $|dV| \le 120\pi(0.1) + 25\pi(0.2) = 12\pi + 5\pi = 17\pi \approx$ **53.4 cm**^{3}.
d. $\dfrac{17\pi}{300\pi} \approx$ **5.7%**. The radius contributes 12π of the 17π, so it is the more important measurement to improve.

Answer to Question 11

a. $P = \dfrac{8.31 \times 300}{0.025} =$ **99 720 Pa**.
b. $\dfrac{\partial P}{\partial T} = \dfrac{R}{V} = 332.4$ Pa/K (pressure rises 332 Pa for each kelvin at fixed volume). $\dfrac{\partial P}{\partial V} = -\dfrac{RT}{V^2} = -3.99 \times 10^6$ Pa/m^3 (pressure falls when volume increases).
c. $dP \approx 332.4(2) + (-3.989 \times 10^6)(-0.001) = 664.8 + 3989 =$ **4 654 Pa**.
d. From $PV = RT$: $\dfrac{\partial P}{\partial V} = -\dfrac{P}{V}$, $\dfrac{\partial V}{\partial T} = \dfrac{R}{P}$, $\dfrac{\partial T}{\partial P} = \dfrac{V}{R}$. The product is $-\dfrac{P}{V} \cdot \dfrac{R}{P} \cdot \dfrac{V}{R} = -1$. It is *not* $+1$ as the cancellation of "fractions" might suggest.

Answer to Question 12

a. Total spending is $3x + 2y$, which must equal 60.
b. $\nabla U = \lambda \nabla g$ gives $(y, x) = \lambda(3, 2)$, so $y = 3\lambda$ and $x = 2\lambda$. Substituting into the budget: $3(2\lambda) + 2(3\lambda) = 12\lambda = 60$, so $\lambda = 5$, $x = 10$, $y = 15$. Satisfaction $U =$ **150**.
c. $\lambda = 5$ means that one extra pound of budget increases the maximum satisfaction by about 5 units (the marginal value of money).
d. With £61: $12\lambda = 61$, $\lambda = \dfrac{61}{12}$, $x = \dfrac{61}{6}$, $y = \dfrac{61}{4}$, $U = \dfrac{61^2}{24} \approx 155.04$. The increase is 5.04, close to the multiplier 5.

Answer to Question 13

a. $r(0, 0) =$ **2 mm** and $r(4, 3) = 2 + 4 + 3 =$ **9 mm**.
b. Inner: $\int_0^3 (2 + x + y)\, dy = 6 + 3x + \dfrac{9}{2} = 10.5 + 3x$. Outer: $\int_0^4 (10.5 + 3x)\, dx = 42 + 24 =$ **66**.
c. $66 \times 1000 =$ **66 000 m**^{3} of water.
d. The district has area 12 km^2, so the mean depth is $\dfrac{66}{12} =$ **5.5 mm**.

Answer to Question 14

a. The region is a disc and the height depends only on $x^2 + y^2 = r^2$. So $V = \displaystyle\int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta$.
b. Inner: $\int_0^2 (4r - r^3)\, dr = \left[2r^2 - \dfrac{r^4}{4}\right]_0^2 = 8 - 4 = 4$. Outer: $2\pi \times 4 = 8\pi \approx 25.1$ **m**^{3}.
c. The maximum height is $z = 4$ m at the centre. The base has radius **2 m**.
d. The base area is $\pi \cdot 2^2 = 4\pi$, so the mean height is $\dfrac{8\pi}{4\pi} =$ **2 m**.

Answer to Question 15

a. $f_x = 3x^2 - 3y = 0$ and $f_y = 3y^2 - 3x = 0$. So $y = x^2$ and $x = y^2$, giving $x = x^4$, that is $x(x^3 - 1) = 0$. Critical points: $(0, 0)$ **and** $(1, 1)$.
b. $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$, so $D = 36xy - 9$.
c. At $(0, 0)$: $D = -9$, a **saddle** (the pass). At $(1, 1)$: $D = 27 > 0$ and $f_{xx} = 6 > 0$, a local **minimum** (the valley) with height -1 m.
d. As $x \to -\infty$ with $y = 0$, $f = x^3 \to -\infty$, so the surface has no global minimum. At

$(-1,-1)$ the value is $-1-1-3=-5$, as in the table, but $(-1,-1)$ is not a critical point: the surface keeps falling beyond it.

Answer to Question 16

a. Minimising $f=x^2+y^2$ is the same as minimising the distance, since distance is the square root.

b. $(2x,2y)=\lambda(3,4)$ gives $x=\dfrac{3\lambda}{2}$, $y=2\lambda$. The constraint: $\dfrac{9\lambda}{2}+8\lambda=25$, so $\dfrac{25\lambda}{2}=25$ and $\lambda=2$. The router goes at $(3,4)$.

c. The distance is $\sqrt{9+16}=$ **5 m**. The formula gives $\dfrac{|25|}{\sqrt{3^2+4^2}}=\dfrac{25}{5}=5$, the same.

d. Maximise x^2+y^2 subject to $x^2+4y^2=25$: $(2x,2y)=\mu(2x,8y)$. Either $x=0$ (then $y=\pm 2.5$, $f=6.25$) or $y=0$ (then $x=\pm 5$, $f=25$). The farthest points are $(\pm 5,0)$, 5 m away.

Answer to Question 17

a. $T=\{(x,y):0\le y\le x\le 1\}$: the triangle below the line $y=x$ and left of $x=1$.

b. The inner integral $\int e^{x^2}dx$ has no elementary antiderivative.

c. In the other order, x goes from 0 to 1 and y from 0 to x:
$I=\int_0^1\int_0^x e^{x^2}\,dy\,dx=\int_0^1 xe^{x^2}dx=\left[\tfrac{1}{2}e^{x^2}\right]_0^1=\dfrac{e-1}{2}\approx 0.859.$

d. Area $=\dfrac{1}{2}$. Average $=\dfrac{I}{1/2}=e-1\approx$ **1.718**.

Answer to Question 18

a. $J^2=\int e^{-x^2}dx\cdot\int e^{-y^2}dy=\iint e^{-x^2}e^{-y^2}dx\,dy=\iint e^{-(x^2+y^2)}dA.$

b. In polar coordinates $x^2+y^2=r^2$ and $dA=r\,dr\,d\theta$:
$\int_0^{2\pi}\int_0^{\infty}e^{-r^2}r\,dr\,d\theta=2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^{\infty}=2\pi\cdot\tfrac{1}{2}=\pi.$

c. $J^2=\pi$ and $J>0$, so $J=\sqrt{\pi}$.

d. Substituting $x=u/\sqrt{2}$ gives $\int e^{-u^2/2}du=\sqrt{2}\int e^{-x^2}dx=\sqrt{2}\sqrt{\pi}=\sqrt{2\pi}$. The density must integrate to 1, so we divide by $\sqrt{2\pi}$.

Answer to Question 19

a. $T_x = 2xy$ and $T_y = x^2$.
b. $\dfrac{dT}{dt} = T_x\dfrac{dx}{dt} + T_y\dfrac{dy}{dt} = 2xy(-2\sin t) + x^2(2\cos t)$. With $x = 2\cos t$, $y = 2\sin t$ this is $-16\cos t\sin^2 t + 8\cos^3 t$.
c. At $t = \frac{\pi}{4}$ both $\cos t$ and $\sin t$ equal $\frac{1}{\sqrt{2}}$. The first term is $-16 \cdot \frac{1}{2\sqrt{2}} = -4\sqrt{2}$ and the second is $8 \cdot \frac{1}{2\sqrt{2}} = 2\sqrt{2}$. The total is $-2\sqrt{2} \approx -2.83$ °**C per minute**: the drone is cooling.
d. $\dfrac{d}{dt}\left(8\cos^2 t\sin t\right) = -16\cos t\sin^2 t + 8\cos^3 t$, the same expression.

Answer to Question 20

a. $u_{xx} + u_{yy} = 2 + (-2) = 0$. For $e^x\cos y$: $u_{xx} = e^x\cos y$ and $u_{yy} = -e^x\cos y$; the sum is 0.
b. $u_x = \dfrac{2x}{x^2+y^2}$, $u_{xx} = \dfrac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \dfrac{2y^2 - 2x^2}{(x^2+y^2)^2}$. By symmetry $u_{yy} = \dfrac{2x^2 - 2y^2}{(x^2+y^2)^2}$. The sum is 0.
c. By the chain rule $v_x = f'(x-ct)$, $v_{xx} = f''$, $v_t = -cf'$, $v_{tt} = c^2 f''$. So $v_{tt} = c^2 v_{xx}$.
d. The wave repeats every 2π in x and every π in t, and moves to the right with speed $c = 2$ (the pattern at time t is the pattern at time 0 shifted by $2t$).