Test answers - Matrices and determinants

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Multiply rows by columns: $PQ = \begin{pmatrix} 2+2 & 8+0 \\ 0+6 & 0+0 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 6 & 0 \end{pmatrix}$ (1) and $QP = \begin{pmatrix} 2+0 & 1+12 \\ 4+0 & 2+0 \end{pmatrix} = \begin{pmatrix} 2 & 13 \\ 4 & 2 \end{pmatrix}$ (1). The products are different.

b. The transpose swaps rows and columns: $(PQ)^T = \begin{pmatrix} 4 & 6 \\ 8 & 0 \end{pmatrix}$ (1), which also equals $Q^T P^T$.

c. $\det(PQ) = 4 \times 0 - 8 \times 6 = -48$. Also $\det P = 6$ and $\det Q = -8$, so $\det P \det Q = -48$, as expected (1).

In (a), $PQ \neq QP$ shows that matrix multiplication is not commutative. In (b), a common error is to write $(PQ)^T = P^T Q^T$; the order reverses. In (c), $\det(QP) = 4 - 52 = -48$ as well.

Answer to Question 2 (4 marks)

Expand along the first row, using the signs $+, -, +$ and the 2×2 minors: $1(1 \times 2 - 4 \times 5) - 2(3 \times 2 - 4 \times 0) + 0 \times (\ldots)$ (2). This gives $1(-18) - 2(6) = -18 - 12 = -30$ (1). Since $\det = -30 \neq 0$, the matrix is invertible (1): its rows are linearly independent, and any system with this coefficient matrix has a unique solution. The zero in the first row saved one minor.

Check with a cofactor expansion down the first column: $1(2 - 20) - 3(4 - 0) + 0 = -30$. A common error is to forget the minus sign on the middle term.

Answer to Question 3 (4 marks)

a. $\det A = 4 \times 2 - 3 \times 3 = 8 - 9 = -1$ (1). Swapping the diagonal entries, negating the others and dividing by the determinant, $A^{-1} = -1 \times \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$ (1).

b. The system is $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+3 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (1). Hence $x = 1$, $y = -1$ (1). Check by substituting into both equations: $4(1) + 3(-1) = 1$ and $3(1) + 2(-1) = 1$. Because $\det A \neq 0$, this is the only solution.

In (a), check: $AA^{-1} = \begin{pmatrix} -8+9 & 12-12 \\ -6+6 & 9-8 \end{pmatrix} = I$. Because the determinant is -1, the inverse has integer entries.

Answer to Question 4 (4 marks)

a. Let £a be the adult price and £c the child price. Then $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 62 \\ 68 \end{pmatrix}$ (1).

b. The determinant is $2 \times 2 - 3 \times 3 = -5$ (1). Replacing the first column by the right-hand side, $a = \dfrac{62 \times 2 - 3 \times 68}{-5} = \dfrac{124 - 204}{-5} = 16$ (1). Replacing the second column, $c = \dfrac{2 \times 68 - 3 \times 62}{-5} = \dfrac{136 - 186}{-5} = 10$ (1). An adult ticket costs £16 and a child ticket £10. Check: $32 + 30 = 62$.

In (b), Cramer's rule applies only because the determinant is non-zero. Check the second equation: $3 \times 16 + 2 \times 10 = 48 + 20 = 68$.

Answer to Question 5 (4 marks)

a. A matrix is singular when its determinant is zero. $\det M = 1 \times 6 - 2k = 6 - 2k$, and $6 - 2k = 0$ gives $k = 3$ (2).

b. With $k = 3$ the equations are $x + 3y = 3$ and $2x + 6y = 6$. The second is exactly twice the first, so the two equations describe the same line and the system has infinitely many solutions (2): $x = 3 - 3y$ for any y. If the right-hand sides were not in the same ratio, there would be no solution.