Worksheet - Matrices and determinants

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Matrix products *(Section A · Skills practice)*

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

a. Compute AB and BA. [4 marks]
b. What does multiplying by B on the right and on the left do to A? [2 marks]
c. Compute A^2. [2 marks]
d. Is $(A+B)^2 = A^2 + 2AB + B^2$ in this case? Explain. [3 marks]

Question 2

Question 2 – Determinants and inverses of 2 by 2 matrices *(Section A · Skills practice)*

Let $C = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

a. Find $\det C$ and C^{-1}. [3 marks]
b. Find $\det D$. What does it tell you about D? [2 marks]
c. Find all values of k for which $\begin{pmatrix} 1 & k \\ k & 4 \end{pmatrix}$ has no inverse. [3 marks]

Question 3

Question 3 – Determinants of 3 by 3 matrices *(Section A · Skills practice)*

Evaluate each determinant, using the method indicated.

a. [redacted], expanding along the first row. [4 marks]

b. [redacted] [2 marks]

c. [redacted] using row operations. [4 marks]

Question 4

Question 4 – Solving systems with matrices *(Section A · Skills practice)*

Solve each system by the method given.

a. $2x + y = 5$ and $5x + 3y = 13$, using the inverse of the coefficient matrix. [4 marks]
b. $2x + y = 5$ and $x + 3y = 10$, using Cramer's rule. [4 marks]

Question 5

Question 5 – Properties of determinants *(Section A · Skills practice)*

Let A be a 3×3 matrix with $\det A = 5$ and B a 3×3 matrix with $\det B = -2$. Find each value, naming the property you use.

a. $\det(AB)$ [2 marks]
b. $\det(2A)$ [2 marks]
c. $\det(A^T)$ and $\det(A^{-1})$ [3 marks]
d. $\det(A^2 B^{-1})$ [3 marks]

Question 6

Question 6 – Transposes, inverses and rotations *(Section A · Skills practice)*

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as before.

a. Verify that $(AB)^T = B^T A^T$. [3 marks]
b. Show that $(AB)^{-1} = B^{-1} A^{-1}$ for these matrices. [3 marks]
c. The matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a rotation through 90°. Find $\det R$ and R^2. [3 marks]
d. Find the area of the triangle with vertices $(0, 0)$, $(4, 1)$ and $(2, 5)$. [3 marks]

Question 7

Question 7 – Prices in the café *(Section B · Real-life practice)*

At a student café, 3 coffees and 2 cakes cost £11.70, and 2 coffees and 3 cakes cost £11.05. Let a coffee cost £c and a cake cost £k.

a. Write the information as a matrix equation $M\mathbf{x} = \mathbf{b}$.
b. Find $\det M$ and M^{-1}.
c. Solve the system.
d. How much do 5 coffees and 4 cakes cost?

Question 8

Question 8 – Weekly takings *(Section B · Real-life practice)*

A small shop sells three products: milk, bread and eggs. The numbers sold on three days and the unit prices (in £) are in the tables.

Day	Milk	Bread	Eggs
Monday	40	25	30
Tuesday	35	30	20
Wednesday	50	20	25

Product	Price £
Milk	1.10
Bread	1.40
Eggs	2.20

a. Write the sales as a 3×3 matrix S and the prices as a column vector $\mathbf{p}$.
b. Compute $S\mathbf{p}$ and interpret each entry.
c. Find the total takings over the three days.
d. Prices rise by 10 per cent. Write the new takings as a matrix product.

Question 9

Question 9 – Moving between city and suburbs *(Section B · Real-life practice)*

Each year, 5% of a city's residents move to the suburbs and 3% of suburban residents move to the city. Let $\mathbf{v}_n = \begin{pmatrix} c_n \\ s_n \end{pmatrix}$ give the populations (in thousands) after n years, so $\mathbf{v}_{n+1} = P\mathbf{v}_n$ with $P = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}$.

a. Check that each column of P sums to 1 and explain why.
b. Initially the city has 600 thousand and the suburbs 400 thousand. Find the populations after one year.
c. Find the populations after 10 years (use a calculator for P^{10}).
d. Find the long-term populations by solving $P\mathbf{v} = \mathbf{v}$ with $c + s = 1000$.

Question 10

Question 10 – Transforming a game character *(Section B · Real-life practice)*

In a video game, a triangular shield has vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$. The game applies the matrix $T = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ to every point.

a. Find the images of the three vertices.

b. Find $\det T$ and use it to find the area of the new shield.
c. Describe T geometrically as a rotation followed by an enlargement.
d. What matrix would undo the transformation?

Question 11

Question 11 – A secret message *(Section B · Real-life practice)*

Agents encode pairs of letters ($A = 0, B = 1, \ldots, Z = 25$) with the matrix $K = \begin{pmatrix} 3 & 3 \\ 2 & 5 \end{pmatrix}$, working modulo 26. The pair $(7, 8)$ stands for "HI".

a. Show that $\det K = 9$ and that $9 \times 3 \equiv 1 \pmod{26}$.
b. Encode "HI".
c. Show that the decoding matrix is $K^{-1} \equiv \begin{pmatrix} 15 & 17 \\ 20 & 9 \end{pmatrix}$ (mod 26).
d. Decode the pair $(19, 2)$ to check.

Question 12

Question 12 – Trail mix *(Section B · Real-life practice)*

A cook mixes oats, nuts and raisins to make 100 g of trail mix. Let the masses be x, y and z grams. Her targets are: total 100 g; "energy points" $2x + 5y + z = 270$; and $x + 3z = 110$.

a. Write the system as $M\mathbf{x} = \mathbf{b}$.
b. Show that $\det M = 5$, so the solution is unique.
c. Solve by row reduction.
d. Check that your answer satisfies all three conditions.

Question 13

Question 13 – Currents in a circuit *(Section B · Real-life practice)*

Kirchhoff's laws for a circuit give the equations $I_1 - I_2 - I_3 = 0$, $2I_1 + 4I_2 = 14$ and $4I_2 - 6I_3 = 2$, where the currents are in amperes.

a. Write the equations as a matrix equation.
b. Evaluate the determinant of the coefficient matrix.
c. Solve for the three currents.
d. Check that the current law holds at the junction.

Question 14

Question 14 – Two connected industries *(Section B · Real-life practice)*

An economy has two industries, steel and energy. To produce 1 unit of steel, 0.2 units

of steel and 0.4 units of energy are used up. To produce 1 unit of energy, 0.3 units of steel and 0.1 units of energy are used up. The consumption matrix is $A = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}$. The outside demand is $\mathbf{d} = (100, 50)$. The production vector $\mathbf{x}$ must satisfy $\mathbf{x} = A\mathbf{x} + \mathbf{d}$.

a. Show that $(I - A)\mathbf{x} = \mathbf{d}$.
b. Find $I - A$ and its determinant.
c. Find $(I - A)^{-1}$ and hence $\mathbf{x}$.
d. Interpret your answer in context: why is the production larger than the demand?

Question 15

Question 15 – Rabbits and matrix powers *(Section C · Problem solving)*

Fibonacci numbers satisfy $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$, $F_1 = 1$. In a rabbit model, F_n is the number of pairs after n months. Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

n	1	2	3	4	5
F_n	1	1	2	3	5

a. Compute Q^2 and Q^3 and compare their entries with the table.
b. Conjecture a formula for Q^n and prove it by induction.
c. Find Q^{10} and read off F_{10}.
d. Find $\det Q^n$ and deduce a relation among F_{n-1}, F_n and F_{n+1} (Cassini's identity).

Question 16

Question 16 – Three-day weather forecast *(Section C · Problem solving)*

A weather model has states Sunny, Cloudy and Rainy. Each column of the transition matrix W gives the probabilities for tomorrow, given today's state: $W =$



.

a. Check that W is a valid transition matrix.
b. Today is sunny, so $\mathbf{v}_0 = (1, 0, 0)^T$. Find $\mathbf{v}_1$ and $\mathbf{v}_2$.
c. Find the long-term proportions by solving $W\mathbf{v} = \mathbf{v}$ with $v_1 + v_2 + v_3 = 1$.
d. On how many days out of 100 will it rain in the long run?

Question 17

Question 17 – Combining transformations *(Section C · Problem solving)*

A graphics engine applies two transformations to every point: first the reflection $S =$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the x-axis, then the rotation $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ through 90° anticlockwise.

a. Find the matrix RS of the combined transformation (first S then R).
b. Find the matrix SR (first R then S) and show that it is different.
c. Find the determinants of S, R and RS. What does a negative determinant mean?
d. Find the image of the point $(3, 1)$ under each combined transformation.

Question 18

Question 18 – Calibrating two sensors *(Section C · Problem solving)*

Two sensors give readings related to an unknown pair (x, y) by $x + ky = 2$ and $kx + 4y = 4$, where k is a calibration constant.

a. Show that the determinant of the coefficient matrix is $4 - k^2$.
b. For which k is there a unique solution? Find it in terms of k.
c. Find what happens when $k = 2$.
d. Find what happens when $k = -2$. Compare with part (c).

Question 19

Question 19 – Volume of a crystal cell *(Section C · Problem solving)*

The edges of a crystal unit cell are given by the vectors $\mathbf{a} = (2, 0, 1)$, $\mathbf{b} = (1, 3, 2)$ and $\mathbf{c} = (1, 1, 2)$ (in ångströms).

a. Write the 3×3 matrix whose rows are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.
b. Find its determinant, and hence the volume of the cell.
c. What is the volume of the tetrahedron with these three edges from one vertex?
d. If the cell is stretched by a factor of 1.5 in every direction, what is the new volume?

Question 20

Question 20 – Challenge: the Cayley-Hamilton theorem *(Section D · Challenge)*

For a 2×2 matrix A the Cayley-Hamilton theorem states $A^2 - (\operatorname{tr} A)A + (\det A)I = 0$. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

a. Find $\operatorname{tr} A$ and $\det A$, and verify the theorem.
b. Use it to express A^{-1} as a combination of A and I.
c. Express A^3 in the form $pA + qI$.
d. Check part (c) by direct multiplication.