Answers - Matrices and determinants

Year: University Year 1

Answer to Question 1

a. Each entry of a product is a row of the first matrix times a column of the second. $AB = \begin{pmatrix} 1\cdot 0 + 2\cdot 1 & 1\cdot 1 + 2\cdot 0 \\ 3\cdot 0 + 4\cdot 1 & 3\cdot 1 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and $BA = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$. They are **different**: matrix multiplication is not commutative.

b. Multiplying by B on the right (AB) swaps the columns of A; multiplying on the left (BA) swaps the rows of A.

c. $A^2 = AA = \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$.

d. No. Expanding carefully, $(A+B)^2 = A^2 + AB + BA + B^2$. The middle terms combine into $2AB$ only if $AB = BA$, and part (a) shows that $AB \neq BA$.

Answer to Question 2

a. $\det C = 3 \times 2 - 1 \times 5 = 1$. For a 2×2 matrix the inverse is $\frac{1}{\det} \times$ (swap the diagonal, negate the other entries), so $C^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$.

b. $\det D = 4 - 4 = 0$, so D is **singular**: it has no inverse, and its rows are proportional.

c. $\det = 4 - k^2 = 0$ gives $k = 2$ **or** -2.

Answer to Question 3

a. $\det = 2(3 \times 2 - 2 \times 1) - 0 + 1(1 \times 1 - 3 \times 1) = 2(4) + (-2) = \mathbf{6}$.

b. The matrix is upper triangular, so the determinant is the product of the diagonal entries: $2 \times 3 \times (-4) = -24$.

c. $R_2 \to R_2 - 4R_1$ gives $(0, -3, -6)$ and $R_3 \to R_3 - 7R_1$ gives $(0, -6, -12)$. Now $R_3 = 2R_2$, so the determinant is **0** (row operations of this type do not change it).

Answer to Question 4

a. The coefficient matrix is $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, with determinant $6 - 5 = 1 \neq 0$, so it is invertible with inverse $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$. Multiplying both sides by the inverse, $\begin{pmatrix} x \\ y \end{pmatrix} =$

$\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 13 \end{pmatrix} = \begin{pmatrix} 15-13 \\ -25+26 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So $x = 2$, $y = 1$. Check: $4 + 1 = 5$ and $10 + 3 = 13$.

b. The determinant of the coefficients is $2 \times 3 - 1 \times 1 = 5$. Cramer's rule replaces one column by the right-hand side: and

. So $x = 1$, $y = 3$. Check: $2 + 3 = 5$ and $1 + 9 = 10$.

Answer to Question 5

a. $\det(AB) = \det A \det B = 5 \times (-2) = -10$ (product rule).

b. Multiplying every row of a 3×3 matrix by 2 multiplies the determinant by 2^3: $\det(2A) = 8 \times 5 = \mathbf{40}$.

c. The determinant is unchanged by transposing: $\det A^T = \mathbf{5}$. And $\det A^{-1} = \dfrac{1}{\det A} = \dfrac{1}{5}$.

d. $\det(A^2B^{-1}) = (\det A)^2 \cdot \dfrac{1}{\det B} = 25 \times \dfrac{1}{-2} = -12.5$.

Answer to Question 6

a. $AB = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ so $(AB)^T = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$. Also $B^TA^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$. They agree.

b. $B^{-1} = B$ (since $B^2 = I$) and $A^{-1} = \dfrac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$. Then $B^{-1}A^{-1} = \dfrac{1}{-2}\begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$. Also $\det(AB) = 2 \times 3 - 1 \times 4 = 2$ and $(AB)^{-1} = \dfrac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$. Dividing by -2 in the first expression gives the same matrix.

c. $\det R = 0 \times 0 - (-1)(1) = 1$, and $R^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$, a rotation through $180°$.

d. Area **9**.

Answer to Question 7

a. Each equation gives one row: $3c + 2k = 11.70$ and $2c + 3k = 11.05$. In matrix form, $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ k \end{pmatrix} = \begin{pmatrix} 11.70 \\ 11.05 \end{pmatrix}$.
b. $\det M = 3 \times 3 - 2 \times 2 = 5 \neq 0$, so M is invertible. Swapping the diagonal entries and negating the others, $M^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$.
c. $\mathbf{x} = M^{-1}\mathbf{b}$ gives $c = \frac{1}{5}(3 \times 11.70 - 2 \times 11.05) = \frac{35.1 - 22.1}{5} =$ **£2.60** and $k = \frac{1}{5}(-2 \times 11.70 + 3 \times 11.05) = \frac{-23.4 + 33.15}{5} =$ **£1.95**. Check: $3 \times 2.60 + 2 \times 1.95 = 7.80 + 3.90 = 11.70$.
d. $5 \times 2.60 + 4 \times 1.95 = 13 + 7.80 =$ **£20.80** for 5 coffees and 4 cakes.

Answer to Question 8

a. Put one day per row and one product per column: $S =$ [redacted], and list the prices in the same product order: $\mathbf{p} =$ [redacted].
b. Each row of S times $\mathbf{p}$ gives (quantity $\times$ price) summed over products: $S\mathbf{p} =$ [redacted] $=$ [redacted]. The entries are the takings on Monday, Tuesday and Wednesday in pounds.
c. Adding the entries, $145 + 124.50 + 138 =$ **£407.50** over the three days.
d. The new price vector is $1.1\,\mathbf{p}$, so the new takings are $S(1.1\,\mathbf{p}) = 1.1\,S\mathbf{p}$. The total becomes $1.1 \times 407.50 =$ **£448.25**.

Answer to Question 9

a. $0.95 + 0.05 = 1$ and $0.03 + 0.97 = 1$: everyone in each place either stays or moves, so the total population is conserved.
b. $\mathbf{v}_1 = P\mathbf{v}_0 = \begin{pmatrix} 0.95(600) + 0.03(400) \\ 0.05(600) + 0.97(400) \end{pmatrix} = \begin{pmatrix} 582 \\ 418 \end{pmatrix}$.
c. $\mathbf{v}_{10} = P^{10}\mathbf{v}_0 \approx \begin{pmatrix} 472.7 \\ 527.3 \end{pmatrix}$ thousand.
d. From $-0.05c + 0.03s = 0$ we get $c = 0.6s$. With $c + s = 1000$: $1.6s = 1000$, $s = 625$ and $c = 375$. In the long run **375 thousand live in the city and 625 thousand in the suburbs**.

Answer to Question 10

a. $T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$; $T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}$; $T\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}-6\\0\end{pmatrix}$. The vertices are $(0,0)$, $(0,4)$ and $(-6,0)$.

b. $\det T = 0 - (-2)(2) = 4$. The original area is $\frac{1}{2} \times 2 \times 3 = 3$, so the new area is $4 \times 3 =$ **12**. Check: the new triangle has base 4 and height 6, area 12.

c. $T = 2\times \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$: an anticlockwise rotation through 90° followed by an enlargement with scale factor 2. The area scale factor is $2^2 = 4$.

d. $T^{-1} = \frac{1}{4}\begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1/2\\-1/2 & 0\end{pmatrix}$.

Answer to Question 11

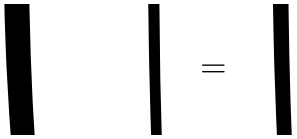
a. $\det K = 3 \times 5 - 3 \times 2 = 15 - 6 = 9$. Also $9 \times 3 = 27 = 26 + 1 \equiv 1 \pmod{26}$, so 3 is the inverse of 9 modulo 26. Because $\det K$ has an inverse mod 26, K can be inverted and messages can be decoded.

b. "HI" is the pair $(7, 8)$. $K\begin{pmatrix}7\\8\end{pmatrix} = \begin{pmatrix}21+24\\14+40\end{pmatrix} = \begin{pmatrix}45\\54\end{pmatrix} \equiv \begin{pmatrix}19\\2\end{pmatrix} \pmod{26}$, which is the pair **"TC"**.

c. The adjugate of K is $\begin{pmatrix}5 & -3\\-2 & 3\end{pmatrix}$. Instead of dividing by 9, multiply by $9^{-1} \equiv 3$: $\begin{pmatrix}15 & -9\\-6 & 9\end{pmatrix} \equiv \begin{pmatrix}15 & 17\\20 & 9\end{pmatrix}$ (mod 26), adding 26 to the negative entries.

d. $\begin{pmatrix}15\cdot 19 + 17\cdot 2\\20\cdot 19 + 9\cdot 2\end{pmatrix} = \begin{pmatrix}319\\398\end{pmatrix} \equiv \begin{pmatrix}7\\8\end{pmatrix}$, since $319 = 12 \times 26 + 7$ and $398 = 15 \times 26 + 8$. This is "HI" again, so the decoding matrix works.

Answer to Question 12

a. The three conditions are $x + y + z = 100$, $2x + 5y + z = 270$ and $x + 3z = 110$. In matrix form,  .

b. Expanding along the first row,
$\det M = 1(15 - 0) - 1(6 - 1) + 1(0 - 5) = 15 - 5 - 5 = 5$. Since $\det M \neq 0$, M is invertible and the system has exactly one solution.

c. $R_2 \to R_2 - 2R_1$ gives $(0, 3, -1 \mid 70)$, and $R_3 \to R_3 - R_1$ gives $(0, -1, 2 \mid 10)$. From the new R_3: $y = 2z - 10$. Substituting into R_2: $3(2z - 10) - z = 70$, so $5z = 100$ and $z = 20$. Then $y = 30$, and from the first row $x = 50$. So **50 g oats, 30 g nuts, 20 g raisins**.

d. $50 + 30 + 20 = 100$; $2(50) + 5(30) + 20 = 270$; $50 + 3(20) = 110$. All three targets are met.

Answer to Question 13

a.  .

b. $\det = 1(-24-0) - (-1)(-12-0) + (-1)(8-0) = -24-12-8 = -44$.

c. From the third equation $I_2 = \dfrac{2+6I_3}{4}$. From the first, $I_1 = I_2 + I_3$. Substitute into the second: $2(I_2+I_3)+4I_2 = 14$, that is $6I_2 + 2I_3 = 14$, so $6\left(\dfrac{2+6I_3}{4}\right) + 2I_3 = 14$, giving $3 + 9I_3 + 2I_3 = 14$ and $I_3 = 1$. Then $I_2 = 2$ and $I_1 = 3$. So $I_1 = 3$ **A,** $I_2 = 2$ **A,** $I_3 = 1$ **A**.

d. $3 - 2 - 1 = 0$. The current entering the junction equals the currents leaving it.

Answer to Question 14

a. Rearrange: $\mathbf{x} - A\mathbf{x} = \mathbf{d}$, so $(I - A)\mathbf{x} = \mathbf{d}$.

b. $I - A = \begin{pmatrix} 0.8 & -0.3 \\ -0.4 & 0.9 \end{pmatrix}$, with determinant $0.72 - 0.12 = 0.6$.

c. $(I-A)^{-1} = \dfrac{1}{0.6}\begin{pmatrix} 0.9 & 0.3 \\ 0.4 & 0.8 \end{pmatrix}$. Then $\mathbf{x} = \dfrac{1}{0.6}\begin{pmatrix} 90+15 \\ 40+40 \end{pmatrix} = \begin{pmatrix} 175 \\ 133.3 \end{pmatrix}$. So steel production is **175 units** and energy **133.3 units**.

d. Each industry also uses the other's output, so it must produce extra to supply the inputs of the other industry (and its own). The extra is $175 - 100 = 75$ units of steel and $133.3 - 50 = 83.3$ of energy.

Answer to Question 15

a. $Q^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $Q^3 = Q^2Q = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. The entries are consecutive Fibonacci numbers: Q^3 contains $F_4 = 3$, $F_3 = 2$ and $F_2 = 1$.

b. Conjecture: $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$. Base case $n = 1$: $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which matches $F_2 = 1$, $F_1 = 1$, $F_0 = 0$. Inductive step: assume it holds for n. Then $Q^{n+1} = Q^nQ = \begin{pmatrix} F_{n+1}+F_n & F_{n+1} \\ F_n + F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$ by the recurrence, so it holds for $n+1$. By induction it holds for all $n \geq 1$.

c. Using the formula, $Q^{10} = \begin{pmatrix} 89 & 55 \\ 55 & 34 \end{pmatrix}$, so $F_{10} = \mathbf{55}$.

d. $\det Q = -1$ and determinants multiply, so $\det Q^n = (-1)^n$. Therefore $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. Check with $n = 10$: $89 \times 34 - 55^2 = 3026 - 3025 = 1$.

Answer to Question 16

a. All entries are non-negative and each column sums to 1 ($0.7+0.2+0.1$, $0.3+0.5+0.2$, $0.1+0.4+0.5$).

b. $\mathbf{v}_1 =$ first column $= (0.7, 0.2, 0.1)^T$.
$\mathbf{v}_2 = W\mathbf{v}_1 = (0.49+0.06+0.01,\ 0.14+0.10+0.04,\ 0.07+0.04+0.05)^T = (0.56, 0.28, 0.16)^T$
.

c. $(W-I)\mathbf{v} = 0$ gives $-0.3a+0.3b+0.1c=0$ and $0.2a-0.5b+0.4c=0$ with $a+b+c=1$. From the first, $c=3a-3b$. Substitute into the second: $0.2a-0.5b+1.2a-1.2b=0$, so $1.4a=1.7b$ and $b=\dfrac{14}{17}a$. Then $c=3a-\dfrac{42}{17}a=\dfrac{9}{17}a$. The sum $a\left(1+\dfrac{14}{17}+\dfrac{9}{17}\right)=\dfrac{40}{17}a=1$ gives $a=\dfrac{17}{40}=0.425$, $b=0.35$, $c=0.225$.

d. Rain occurs on **22.5 days in 100** in the long run.

Answer to Question 17

a. $RS=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$: reflection in the line $y=x$.

b. $SR=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$: reflection in the line $y=-x$. This differs from RS.

c. $\det S=-1$, $\det R=1$, $\det RS=-1$. A negative determinant means that orientation is reversed (a reflection) and the area is multiplied by $|\det|=1$.

d. Under RS: $\binom{0\cdot3+1\cdot1}{1\cdot3+0\cdot1}=\binom{1}{3}$. Under SR: $\binom{-1}{-3}$.

Answer to Question 18

a. The matrix is $\begin{pmatrix}1&k\\k&4\end{pmatrix}$, with determinant $1\times4-k\times k=4-k^2$.

b. A unique solution exists when $k\neq\pm2$. By Cramer's rule:
$x=\dfrac{2\times4-k\times4}{4-k^2}=\dfrac{4(2-k)}{(2-k)(2+k)}=\dfrac{4}{2+k}$ and
$y=\dfrac{1\times4-k\times2}{4-k^2}=\dfrac{2(2-k)}{(2-k)(2+k)}=\dfrac{2}{2+k}$.

c. For $k=2$ the equations are $x+2y=2$ and $2x+4y=4$. The second is twice the first, so there are **infinitely many solutions** (all points on the line $x+2y=2$). This matches $\dfrac{4}{2+k}\to1$ only formally; the formulas break down.

d. For $k=-2$ the equations are $x-2y=2$ and $-2x+4y=4$. The second says $x-2y=-2$, which contradicts the first, so there is **no solution**. The

determinant is zero in both cases, but the right-hand sides decide whether the system is consistent.

Answer to Question 19

a.  .

b. The determinant is 6 (computed earlier), so the volume of the parallelepiped is **6 Å^3**.

c. A tetrahedron has $\frac{1}{6}$ of the parallelepiped's volume: **1 Å^3**.

d. Volumes scale by the cube of the linear factor: $6 \times 1.5^3 = 6 \times 3.375 = \mathbf{20.25}$ **Å^3**. Equivalently, the matrix $1.5\,I_3$ has determinant 1.5^3.

Answer to Question 20

a. $\operatorname{tr} A = 1 + 4 = 5$ and $\det A = 4 - 6 = -2$, so the theorem says $A^2 - 5A - 2I = 0$. Indeed $A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ and $5A = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$, so $A^2 - 5A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$, and the theorem holds.

b. Rearrange: $A^2 - 5A = 2I$, so $A(A - 5I) = 2I$. Dividing by 2, $A \cdot \frac{1}{2}(A - 5I) = I$, hence $A^{-1} = \frac{1}{2}(A - 5I) = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

c. From $A^2 = 5A + 2I$, multiply by A:
$A^3 = 5A^2 + 2A = 5(5A + 2I) + 2A = 27A + 10I$. So $p = 27$ and $q = 10$.

d. $27A + 10I = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$. Directly, $A^3 = A \cdot A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$. They agree.