Test answers - Logic, sets and proof techniques

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. There exists a real x such that for every real y, $y \leq x$ (2).
b. If n is not even (odd), then n is not divisible by 6 (2).

In (a), negating swaps each quantifier, $\forall$ becomes $\exists$ and $\exists$ becomes $\forall$, and the final statement $y > x$ becomes $y \leq x$, not $y < x$. The original statement is true, so its negation is false: no real number is greater than or equal to every real number. In (b), the contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$, which is logically equivalent to the original. A common error is to give the converse, if n is even then n is divisible by 6, which is false (take $n = 4$).

Answer to Question 2 (4 marks)

a. At least one language: $50 - 10 = 40$ (1). $|F \cup S| = 30 + 22 - |F \cap S| = 40$ (1), so both $= 12$ (1).
b. Exactly one: $40 - 12 = 28$ (1).

In (a), the 10 people who speak neither language are outside both sets, so $|F \cup S| = 40$. The inclusion-exclusion principle corrects for counting the people in $F \cap S$ twice when adding 30 and 22. Solving $52 - |F \cap S| = 40$ gives 12. In (b), French only is $30 - 12 = 18$ and Spanish only is $22 - 12 = 10$, and $18 + 10 = 28$. Check with a Venn diagram: $18 + 12 + 10 + 10 = 50$ people in total.

Answer to Question 3 (4 marks)

Let the integers be n, $n + 1$, $n + 2$ (1). Their sum is $3n + 3$ (2), which is $3(n + 1)$, divisible by 3 (1).

This is a direct proof: we start from the general form of three consecutive integers and reach the conclusion by algebra. Since n is an integer, $n + 1$ is an integer, so $3(n + 1)$ is a multiple of 3 by definition of divisibility. Checking examples such as $4 + 5 + 6 = 15$ illustrates the result but proves nothing. A neater choice is $n - 1$, n, $n + 1$, whose sum is $3n$ directly.

Answer to Question 4 (4 marks)

Suppose there is a largest prime and list all primes $p_1, \ldots, p_k$ (1). Let $N = p_1 p_2 \cdots p_k + 1$ (1). N leaves remainder 1 when divided by each p_i, so no p_i divides N (1). But $N > 1$ has a prime factor, which is a prime not in the list: contradiction (1).

The proof does not claim that N itself is prime: for example $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\,031 = 59 \times 509$. It only shows that some prime outside the finite list must exist.

Answer to Question 5 (4 marks)

Base $n = 1$: $1 = 1^2$ (1). Assume $\sum_{i=1}^{k}(2i-1) = k^2$ (1). Then $\sum_{i=1}^{k+1}(2i-1) = k^2 + 2(k+1) - 1 = k^2 + 2k + 1$ (1) $= (k+1)^2$ (1).

The base case checks that the statement holds for the first value. The inductive hypothesis assumes the statement for some $k \geq 1$. In the inductive step, the sum up to $k+1$ is the sum up to k plus the next term $2(k+1) - 1 = 2k + 1$, and the hypothesis replaces the first part with k^2. Since $k^2 + 2k + 1 = (k+1)^2$, the statement holds for $k+1$. By the principle of mathematical induction it holds for all $n \geq 1$: the first n odd numbers always add up to n^2, for example $1 + 3 + 5 + 7 = 16$.