Test answers - Integration and the fundamental theorem of calculus

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Use integration by parts, choosing $u = x$ (it becomes simpler when differentiated) and $v' = \cos x$, so $u' = 1$ and $v = \sin x$ (1). Then $\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$ (1). Differentiating gives back $x \cos x$.

b. Let $u = 1 + x^2$, so $du = 2x \, dx$; the limits become 1 and 2. The integral is $\frac{1}{2} \int_1^2 \frac{du}{u}$ (1) $= \frac{1}{2}[\ln u]_1^2 = \frac{1}{2} \ln 2$ (1), about 0.347.

In (a), choosing $u = \cos x$ instead makes the new integral $\int \frac{x^2}{2} \sin x \, dx$ harder, so the choice of u matters.

Answer to Question 2 (4 marks)

First split into partial fractions: $1 = A(x+1) + B(x-1)$; $x = 1$ gives $A = \frac{1}{2}$ and $x = -1$ gives $B = -\frac{1}{2}$, so $\dfrac{1}{(x-1)(x+1)} = \dfrac{1}{2}\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right)$ (2). Integrating each term and combining the logarithms gives $\frac{1}{2}\left[\ln \dfrac{x-1}{x+1}\right]_2^3$ (1). Substituting the limits, $\frac{1}{2}\left(\ln \dfrac{2}{4} - \ln \dfrac{1}{3}\right) = \frac{1}{2}\left(\ln \dfrac{1}{2} - \ln \dfrac{1}{3}\right) = \frac{1}{2} \ln \dfrac{3}{2}$ (1), as a single logarithm. Since $\frac{3}{2} > 1$, the answer is positive (about 0.203), as expected for a positive integrand on $[2, 3]$.

A common error is to integrate $\frac{1}{(x-1)(x+1)}$ directly as $\ln|(x-1)(x+1)|$; partial fractions are needed first. The integrand is continuous on $[2, 3]$, so the integral is proper.

Answer to Question 3 (4 marks)

a. The area under $y = x^2$ from 0 to 2 is $\int_0^2 x^2 dx = \left[\dfrac{x^3}{3}\right]_0^2 = \dfrac{8}{3}$ square units (1).

b. Rotating about the x-axis, each slice is a disc of radius $y = x^2$, so $V = \pi \int_0^2 y^2 dx = \pi \int_0^2 x^4 dx$ (1) $= \pi \left[\dfrac{x^5}{5}\right]_0^2 = \dfrac{32\pi}{5} \approx 20.1$ cubic units (1).

c. The mean value is $\dfrac{1}{b-a} \int_a^b y \, dx = \dfrac{1}{2} \cdot \dfrac{8}{3} = \dfrac{4}{3}$ (1). A rectangle of height $\frac{4}{3}$ on $[0, 2]$ has the same area as R.

In (b), a common error is to use $\pi \int_0^2 x^2\,dx$; squaring the radius is essential because each disc has area πy^2. In (c), dividing by the interval length 2 is the key step.

Answer to Question 4 (4 marks)

a. Replace the infinite limit by b and take a limit: $\int_1^b x^{-3/2}dx = \left[-2x^{-1/2}\right]_1^b = 2 - \dfrac{2}{\sqrt{b}}$ (1). As $b \to \infty$, $\dfrac{2}{\sqrt{b}} \to 0$, so the integral converges to **2** (1).

b. Similarly $\int_1^b x^{-1/2}dx = \left[2\sqrt{x}\right]_1^b = 2\sqrt{b} - 2$ (1). This tends to ∞ as $b \to \infty$, so the integral diverges (1). In general $\int_1^\infty x^{-p}dx$ converges only when $p > 1$; here $p = \frac{1}{2}$, so the area under the curve is infinite even though $\frac{1}{\sqrt{x}} \to 0$.

Answer to Question 5 (4 marks)

a. Let $F(u) = \int_1^u \ln t\,dt$, so $F'(u) = \ln u$ by the fundamental theorem. By the chain rule, $\dfrac{d}{dx}F(x^3) = \ln(x^3) \cdot 3x^2$ (1) $= 3\ln x \cdot 3x^2 = 9x^2 \ln x$ (1).

b. The width of each strip is $\frac{1}{4}$, and the right endpoints are $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The sum is $\frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right)$ (1) $= \frac{1}{4} \cdot \frac{30}{16} = \frac{15}{32} \approx 0.469$ (1). It overestimates the exact value $\frac{1}{3}$, because x^2 is increasing.

In (a), a common error is to give only $\ln(x^3)$, forgetting the chain rule factor $3x^2$ from the upper limit. In (b), the error of the estimate is $\frac{15}{32} - \frac{1}{3} = \frac{13}{96} \approx 0.135$.