Lesson - Integration and the fundamental theorem of calculus

Year: University Year 1

Integration accumulates quantities such as area and volume. At university the definite integral is defined through Riemann sums, and the fundamental theorem of calculus links it to differentiation.

Starter activity

Draw the graph of $y = x^2$ for $0 \le x \le 1$ on squared paper and count squares to estimate the area underneath. Now estimate again with 4 thin rectangles whose heights are the values of y at their left edges, and with 4 rectangles whose heights come from their right edges. Which estimate is too small and which is too large? What do you think happens to the two estimates if you use 100 rectangles?

The definite integral as a limit of sums

Definition: Riemann integral

Let f be bounded on $[a, b]$ and $P = \{a = x_0$ a partition. The **lower and upper Riemann sums** are $L(f, P) = \sum m_i \Delta x_i$ and $U(f, P) = \sum M_i \Delta x_i$, where m_i and M_i are the infimum and supremum of f on the ith subinterval. f is **Riemann integrable** if the supremum of lower sums equals the infimum of upper sums; the common value is $\int_a^b f(x)\,dx$.

Property: continuous functions

Every continuous function on $[a, b]$ is Riemann integrable.

Example 1

Estimate $\int_0^1 x^2 dx$ with four equal strips using left endpoints.

$\frac{1}{4}\left(0 + \frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right) = \frac{7}{32} \approx 0.219$. This underestimates the true value $\frac{1}{3}$ because f is increasing.

The fundamental theorem of calculus

Theorem: fundamental theorem of calculus

Part 1. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is differentiable and $F'(x) = f(x)$.

Part 2. If $G' = f$ on $[a, b]$ then $\int_a^b f(x)\,dx = G(b) - G(a)$.

Example 2

Find $\dfrac{d}{dx}\displaystyle\int_0^{x^2} e^{t^2}\,dt$**.**

By Part 1 and the chain rule, the derivative is $e^{(x^2)^2} \cdot 2x = 2x\,e^{x^4}$.

Standard integrals, substitution and parts

Property: standard integrals

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$, $\int \frac{1}{x} dx = \ln|x| + C$, $\int e^x dx = e^x + C$,
$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C$, $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$.

Method: substitution

1. Choose $u = g(x)$ so that $g'(x)\,dx$ appears.
2. Rewrite the integral as $\int f(u)\,du$.
3. For a definite integral change the limits to u-values.
4. Integrate, and substitute back if the integral is indefinite.

Example 3

Evaluate $\displaystyle\int \frac{2x}{x^2+1}dx$ **and** $\displaystyle\int_0^1 xe^{x^2}dx$**.**

With $u = x^2 + 1$: $\ln(x^2+1) + C$. With $u = x^2$ and new limits 0 and 1: $\frac{1}{2}\int_0^1 e^u du = \frac{e-1}{2}$.

Rule: integration by parts

$\displaystyle\int u\,dv = uv - \int v\,du$. Choose u as the factor that becomes simpler when differentiated (logs, polynomials), and dv as the factor you can integrate.

Example 4

Find $\displaystyle\int xe^x dx$ **and** $\displaystyle\int e^x \cos x\,dx$**.**

First: $xe^x - \int e^x dx = (x-1)e^x + C$. Second: apply the formula twice and solve for the integral to get $\frac{e^x(\sin x + \cos x)}{2} + C$.

Partial fractions and trigonometric integrals

Method: rational and trigonometric integrands

1. For a rational function with a factorisable denominator, split into partial fractions and integrate each term.
2. For even powers of $\sin$ and $\cos$, use $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.
3. For an odd power, peel off one factor and use $\sin^2 + \cos^2 = 1$.

Example 5

Find $\int \frac{dx}{x^2-1}$ and $\int_0^{\pi} \sin^3 x\, dx$.

$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$, so the integral is $\frac{}{}$. And $\int_0^{\pi}(1-\cos^2 x)\sin x\, dx = \frac{4}{3}$.

Improper integrals

Rule: improper integrals

An integral over an infinite interval, or of an unbounded function, is defined as a limit. $\int_1^{\infty} \frac{dx}{x^p}$ converges to $\frac{1}{p-1}$ if $p > 1$ and diverges if $p \leq 1$.

Example 6

Evaluate $\int_0^1 \ln x\, dx$ and $\int_0^{\infty} x^n e^{-x} dx$.

$\lim_{\epsilon \to 0^+}[x\ln x - x]_{\epsilon}^1 = -1$, and $\int_0^{\infty} x^n e^{-x} dx = n!$.

Applications

Property: area, volume and mean value

- Area between curves with $f \geq g$: $\int (f-g)dx$.
- Volume of revolution about the x-axis: $\pi \int y^2 dx$.
- Mean value of f on $[a,b]$: $\frac{1}{b-a}\int_a^b f$.

Example 7

Find the area between $y = x$ and $y = x^2$, and the volume when $y = \sqrt{x}$, $0 \leq x \leq 4$, is rotated about the x-axis.

$\int_0^1 (x - x^2)dx = \frac{1}{6}$. And $\pi \int_0^4 x\,dx = 8\pi$.

Watch out

- Forgetting $+C$, or forgetting to change limits in a substitution.
- Choosing u and dv the wrong way round.
- Writing $\int \frac{1}{x}dx = \ln x$ without the modulus.

Technique and understanding

Method: choosing an integration technique

1. Try to simplify the integrand: expand brackets, use identities, divide polynomials.
2. Look for an inner function whose derivative also appears: use substitution.
3. For a product of unrelated functions, such as a polynomial times e^x, $\sin x$ or $\ln x$, use integration by parts and choose u so that u' is simpler (the order LIATE: logarithm, inverse trigonometric, algebraic, trigonometric, exponential, can help).
4. For a rational function, factorise the denominator and use partial fractions.
5. Differentiate your answer to check it.

Worked example 8: Substitution with changed limits

Evaluate $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$**.**

Let $u = x + 1$, so $x = u - 1$ and $dx = du$. The limits change from $x = 0, 3$ to $u = 1, 4$. Then the integral becomes $\int_1^4 \frac{u-1}{\sqrt{u}}du = \int_1^4 \left(u^{1/2} - u^{-1/2}\right)du = \left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^4$. At $u = 4$ the bracket equals $\frac{16}{3} - 4 = \frac{4}{3}$, and at $u = 1$ it equals $\frac{2}{3} - 2 = -\frac{4}{3}$. So the value is $\frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$. Changing the limits means you never need to go back to x.

Worked example 9: Integration by parts twice

Find $\displaystyle\int e^x \cos x\,dx$**.**

Call the integral I. First take $u = \cos x$, $v' = e^x$, so $I = e^x \cos x + \int e^x \sin x\,dx$. Apply parts again to the new integral, with $u = \sin x$:
$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. Substituting,
$I = e^x \cos x + e^x \sin x - I$, so $2I = e^x(\sin x + \cos x)$ and $I = \frac{1}{2}e^x(\sin x + \cos x) + C$.
The original integral reappears, and we solve for it like an equation.

Worked example 10: Improper integral with a singularity

Evaluate $\int_0^1 \ln x \, dx$.

The integrand is undefined at $x = 0$, so write the integral as $\lim_{a \to 0^+} \int_a^1 \ln x \, dx$. An antiderivative is $x \ln x - x$, so the integral equals $[x \ln x - x]_a^1 = -1 - a \ln a + a$. As $a \to 0^+$ we have $a \ln a \to 0$ (a standard limit), so the limit is -1. The integral converges even though the integrand is unbounded: the area is finite.

Worked example 11: The area between two curves

Find the area enclosed by $y = x^2$ and $y = 2x + 3$.

First find where the curves meet: $x^2 = 2x + 3$ gives $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$, so $x = -1$ and $x = 3$. Between these values the line lies above the parabola, so the area is

$\int_{-1}^{3} (2x + 3 - x^2) dx = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^{3} = (9 + 9 - 9) - (1 - 3 + \frac{1}{3}) = 9 + \frac{5}{3} = \frac{32}{3}$.

Always sketch the curves first so that you subtract in the right order.

Worked example 12: Volume of a cone by rotation

Derive the volume of a cone of radius r and height h.

Rotate the line $y = \dfrac{r}{h} x$ for $0 \le x \le h$ about the x-axis. A disc at position x has radius $\dfrac{r}{h} x$, so $V = \pi \int_0^h \dfrac{r^2}{h^2} x^2 dx = \dfrac{\pi r^2}{h^2} \cdot \dfrac{h^3}{3} = \dfrac{1}{3} \pi r^2 h$. This is the formula you learned at school, now proved by integration.

Worked example 13: Mean value of a function

Find the mean value of $f(x) = \sin^2 x$ on $[0, \pi]$.

The mean value is $\dfrac{1}{\pi} \int_0^{\pi} \sin^2 x \, dx$. Using $\sin^2 x = \dfrac{1 - \cos 2x}{2}$, the integral is $\left[\dfrac{x}{2} - \dfrac{\sin 2x}{4} \right]_0^{\pi} = \dfrac{\pi}{2}$. So the mean value is $\dfrac{1}{2}$. This is why the average power in an alternating current circuit is half of the peak power.

Key fact: why the fundamental theorem is true

Let $A(x)$ be the area under f from a to x. When x increases by a small amount h, the area grows by a thin strip whose width is h and whose height is close to $f(x)$. So $A(x + h) - A(x) \approx f(x) \, h$, and dividing by h gives $\dfrac{A(x + h) - A(x)}{h} \approx f(x)$. In the

limit $h \to 0$ this says $A'(x) = f(x)$: the area function is an antiderivative of f. Any two antiderivatives differ by a constant, which cancels when we subtract $F(a)$ from $F(b)$. This is why the process of finding an area, which looks like an infinite sum of tiny pieces, turns into a short subtraction.

Key fact: numerical integration

When no antiderivative can be found, or when the function is known only from measurements, we approximate the area. The trapezium rule joins consecutive data points with straight lines and adds the areas of the trapezia. Simpson's rule joins groups of three points with parabolas, so it needs an even number of strips but is usually much more accurate for smooth curves. Halving the strip width reduces the error of the trapezium rule by a factor of about four, and the error of Simpson's rule by a factor of about sixteen. In practice engineers choose the rule according to the smoothness of the data and the accuracy they need, and they always compare two estimates with different numbers of strips.

Key fact: comparing improper integrals

The family $\int_1^\infty x^{-p}dx$ is the benchmark for every improper integral with an infinite range. It converges when $p > 1$, where it equals $\dfrac{1}{p-1}$, and it diverges when $p \le 1$. To test a more complicated integrand, compare it with a benchmark. If $0 \le f(x) \le g(x)$ for large x and the integral of g converges, then the integral of f converges too, because its area is smaller. If instead f is larger than a divergent benchmark, then it diverges. For integrands that blow up at a point, such as $\dfrac{1}{\sqrt{x}}$ near zero, the benchmark is $\int_0^1 x^{-p}dx$, which behaves the other way round and converges when p. Learning to recognise these two families saves a great deal of calculation.

Note: indefinite and definite integrals

An indefinite integral $\int f(x)\,dx$ is a family of functions (the antiderivatives), written with a constant $+C$. A definite integral $\int_a^b f(x)\,dx$ is a single number, the signed area. The fundamental theorem connects them: if $F' = f$ then $\int_a^b f = F(b) - F(a)$. This number is negative where the graph is below the axis, so area problems often need a split at the roots.

Watch out: signs and limits

- Forgetting to change the limits (or to go back to x) after a substitution.
- Dropping the minus sign when integrating $\sin$ or $\cos$: $\int \sin x\,dx = -\cos x$.
- Treating the area under a graph as $\int f$ when f changes sign.

- Writing $\int \frac{dx}{x} = \ln x$ without the modulus when x may be negative.
- Leaving out the constant of integration in an indefinite integral, or adding one to a definite integral, where it always cancels.
- Forgetting that a quick sketch of the region usually shows which curve is above the other and where the limits are.

Did you know?

Archimedes found the area of a parabolic segment in the third century BC by a method of exhaustion that is very close to modern integration. Nearly two thousand years later Newton and Leibniz realised that finding areas and finding tangents are inverse operations, which is the fundamental theorem of calculus. Bernhard Riemann gave the precise modern definition of the integral in 1854.

Otto's tip

Always differentiate your answer to check an indefinite integral.