Worksheet - Integration and the fundamental theorem of calculus

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Basic integrals *(Section A · Skills practice)*

Find each indefinite integral, or evaluate each definite integral exactly.

a. $\int (3x^2 - 2x + 1)\,dx$ [2 marks]

b. $\int_0^2 (x^2 + 1)\,dx$ [2 marks]

c. $\int_1^4 \sqrt{x}\,dx$ [3 marks]

d. $\int \left(e^{2x} + \dfrac{1}{x}\right) dx$ for $x > 0$ [2 marks]

Question 2

Question 2 – Substitution *(Section A · Skills practice)*

Use a substitution to find or evaluate each integral.

a. $\int \dfrac{2x}{x^2 + 1}\,dx$ [2 marks]

b. $\int_0^1 x e^{x^2}\,dx$ [3 marks]

c. $\int \cos x \sin^3 x\,dx$ [3 marks]

Question 3

Question 3 – Integration by parts *(Section A · Skills practice)*

Use integration by parts, $\int u\,v'\,dx = uv - \int u'v\,dx$, to find each integral.

a. $\int x e^x\,dx$ [3 marks]

b. $\int x^2 \ln x\,dx$ [4 marks]

c. $\int x \sin x \, dx$ [3 marks]

d. $\int \ln x \, dx$ [3 marks]

Question 4

Question 4 – Partial fractions *(Section A · Skills practice)*

Express the integrand in partial fractions and integrate.

a. $\int \frac{dx}{x^2 - 1}$ for $x > 1$ [4 marks]

b. $\int \frac{3x + 5}{(x + 1)(x + 2)} \, dx$ for $x > -1$ [4 marks]

Question 5

Question 5 – Trigonometric integrals *(Section A · Skills practice)*

Find or evaluate each integral using a trigonometric identity or substitution.

a. $\int \sin^2 x \, dx$ [3 marks]

b. $\int_0^{\pi} \sin^3 x \, dx$ [4 marks]

c. $\int \frac{dx}{x^2 + 4}$ [3 marks]

Question 6

Question 6 – The fundamental theorem and Riemann sums *(Section A · Skills practice)*

Answer each part.

a. Find $\frac{d}{dx} \int_0^{x^2} e^{t^2} dt$. [3 marks]

b. Find $\frac{d}{dx} \int_x^3 \sin(t^2) \, dt$. [2 marks]

c. Compute the left-endpoint Riemann sum with $n = 4$ for $\int_0^1 x^2 dx$ and compare with the exact value. [4 marks]

Question 7

Question 7 – Distance travelled *(Section B · Real-life practice)*

A jogger runs along a straight track. Her velocity in m/s after t seconds is $v(t) = t^2 - 4t + 3$ for $0 \le t \le 4$ (negative means running backwards).

a. Find her displacement from the start after 4 seconds.
b. Find when she is stationary.
c. Find the total distance she runs in the 4 seconds.
d. Explain why displacement and distance differ.

Question 8

Question 8 – Filling a tank *(Section B · Real-life practice)*

Water flows into a tank at a rate $r(t) = 12e^{-0.1t}$ litres per minute, where t is the time in minutes. The tank starts empty.

a. Write down the volume $V(t)$ as an integral and evaluate it.
b. Find the volume after 30 minutes.
c. Find the time at which the tank holds 100 litres.
d. What is the largest volume the tank can ever hold with this inflow?

Question 9

Question 9 – Average temperature *(Section B · Real-life practice)*

The temperature in a school hall, in °C, is modelled by $T(t) = 12 - 6\cos\dfrac{\pi t}{12}$ for $0 \le t \le 24$, where t is the number of hours after midnight.

a. Find the mean temperature over the whole day.
b. Find the mean temperature between 06:00 and 18:00 using $\dfrac{1}{b-a}\int_a^b T\,dt$.
c. Why is the answer to (b) higher than (a)?
d. At what time of day is the temperature equal to the mean found in (a)?

Question 10

Question 10 – A field between a road and a stream *(Section B · Real-life practice)*

On a map the road is the line $y = x$ and the stream is the curve $y = x^2$, with lengths in units of 100 m. The two meet at two points and enclose a field.

a. Find the points where the road and stream meet.
b. Find the area of the field in square units and in m^2.
c. Grass seed costs 40p per m^2 of field. Find the cost to the nearest £10.
d. The farmer wants fencing along the road side only. Find that length in metres.

Question 11

Question 11 – A vase by rotation *(Section B · Real-life practice)*

A decorative vase is made by rotating the curve $y = \sqrt{x}$ for $0 \le x \le 16$ (lengths in cm) about the x-axis.

a. Show that the volume is $V = \pi \int_0^{16} x\,dx$ and evaluate it.
b. Express the volume in millilitres (1 cm^3 = 1 ml).
c. The vase is filled with sand of density 1.6 g/ml. Find the mass of sand.
d. Find the radius of the vase at the open end.

Question 12

Question 12 – Work done stretching a spring *(Section B · Real-life practice)*

A spring in a physics laboratory exerts a force $F = 40x$ newtons when it is extended by x metres. The work done in extending it from a to b is $\displaystyle\int_a^b F\,dx$.

a. Find the work done stretching from natural length to 0.3 m.
b. Find the work done stretching from 0.3 m to 0.5 m.
c. Compare the two results and explain why the second is larger even though the extension is only 0.2 m.
d. How far can the spring be stretched with 10 J of work from natural length?

Question 13

Question 13 – Waiting for the bus *(Section B · Real-life practice)*

The waiting time X (in minutes) of a passenger at a rural bus stop has density $f(x) = \dfrac{1}{4}xe^{-x/2}$ for $x \ge 0$. You are given $\displaystyle\int_0^{\infty} xe^{-x/2}dx = 4$.

a. Verify that $\displaystyle\int_0^{\infty} f(x)\,dx = 1$.
b. Find the probability of waiting less than 2 minutes: show that $P(X$.
c. Find the mean waiting time $\displaystyle\int_0^{\infty} x f(x)\,dx$.
d. A passenger arrives. Is she more likely to wait less than 2 minutes or more than 2 minutes?

Question 14

Question 14 – Total dose of a drug *(Section B · Real-life practice)*

A drug is absorbed so that the rate of release into the blood is $r(t) = 50e^{-0.2t}$ mg per hour for $t \ge 0$ (in hours).

a. Find the amount released in the first 5 hours.
b. Find the total amount that will ever be released.
c. What percentage of the total is released in the first 5 hours?
d. After how many hours has 90% of the total been released?

Question 15

Question 15 – Area of a river cross-section *(Section C · Problem solving)*

To estimate the flow of a river, a surveyor measures the depth every 2 metres across a 12 m wide channel.

Distance across (m)	0	2	4	6	8	10	12
Depth (m)	0	0.6	1.1	1.5	1.2	0.7	0

a. Use the trapezium rule with 6 strips to estimate the cross-sectional area.
b. Use Simpson's rule with 6 strips: $\frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6]$.
c. The water flows at an average of 0.8 m/s. Find the flow in m^3/s from each method.
d. Which estimate is likely to be better, and why?

Question 16

Question 16 – Growth of a lake population *(Section C · Problem solving)*

The number of fish P in a lake satisfies $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{100}\right)$ with t in years. The time to grow from P_1 to P_2 is $T = \int_{P_1}^{P_2} \frac{dP}{0.1P(1 - P/100)}$.

a. Show that $\frac{1}{0.1P(1 - P/100)} = 10\left(\frac{1}{P} + \frac{1}{100 - P}\right)$.
b. Evaluate the integral to show that $T = 10\left[\ln\frac{P}{100 - P}\right]_{P_1}^{P_2}$.
c. Find the time to grow from 20 fish to 50 fish.
d. Explain why the time to reach $P_2 = 100$ is infinite.

Question 17

Question 17 – Gabriel's horn *(Section C · Problem solving)*

Rotate the curve $y = \frac{1}{x}$ for $x \geq 1$ about the x-axis to form an infinitely long horn. A painter asks whether the horn can be filled with paint and whether its surface can be painted.

a. Show that the volume is $V = \pi \displaystyle\int_1^{\infty} \frac{dx}{x^2}$ and evaluate it.
b. The surface area is $S = 2\pi \displaystyle\int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$. Show that $S \geq 2\pi \displaystyle\int_1^{\infty} \frac{dx}{x}$.
c. Evaluate the integral in (b) and deduce that S is infinite.
d. Explain the apparent paradox.

Question 18

Question 18 – Lifetimes of components *(Section C · Problem solving)*

In reliability theory, the lifetime of a component can be modelled by $I_n = \displaystyle\int_0^{\infty} t^n e^{-t} dt$, where $n \geq 0$ is an integer.

a. Evaluate I_0.
b. Use integration by parts to show that $I_n = nI_{n-1}$ for $n \geq 1$.
c. Hence find I_1, I_2, I_3 and I_4, and guess a formula for I_n.
d. Explain why $\lim_{t\to\infty} t^n e^{-t} = 0$ is needed in part (b).

Question 19

Question 19 – Filling a bowl *(Section C · Problem solving)*

A bowl is formed by rotating $y = x^2$ for $0 \leq y \leq 9$ about the y-axis (lengths in cm). It is filled from a tap at 5 cm^3 per second.

a. Show that the volume of water of depth h is $V = \dfrac{\pi}{2}h^2$.
b. Find the capacity of the bowl in ml.
c. How long does it take to fill the bowl?
d. Find the depth when the bowl is half full, and the rate at which the depth is rising at that moment.

Question 20

Question 20 – Challenge: an integral equation *(Section D · Challenge)*

A continuous function f satisfies $f(x) = 1 + \displaystyle\int_0^{x} f(t)\,dt$ for all $x \geq 0$.

a. Use the fundamental theorem to show that $f'(x) = f(x)$, and state $f(0)$.
b. Solve the differential equation and find f.
c. Check your answer by substituting back into the integral equation.
d. Starting with $f_0(x) = 1$ and $f_{n+1}(x) = 1 + \int_0^x f_n(t)dt$, compute f_1, f_2, f_3. What do they tend to?