Answers - Integration and the fundamental theorem of calculus

Year: University Year 1

Answer to Question 1

a. Integrate term by term with the power rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$: $\int 3x^2\,dx = x^3$, $\int 2x\,dx = x^2$ and $\int 1\,dx = x$. So the answer is $x^3 - x^2 + x + C$. Differentiating gives back $3x^2 - 2x + 1$, which checks it.

b. An antiderivative is $\dfrac{x^3}{3} + x$. By the fundamental theorem,

$$\left[\frac{x^3}{3} + x\right]_0^2 = \left(\frac{8}{3} + 2\right) - 0 = \frac{14}{3}.$$

c. Write $\sqrt{x} = x^{1/2}$, so an antiderivative is $\dfrac{2}{3}x^{3/2}$. Then $\left[\dfrac{2}{3}x^{3/2}\right]_1^4 = \dfrac{2}{3}(8 - 1) = \dfrac{14}{3}$, since $4^{3/2} = 8$.

d. Since $\dfrac{d}{dx}e^{2x} = 2e^{2x}$, we have $\int e^{2x}dx = \dfrac{1}{2}e^{2x}$; and $\int \dfrac{1}{x}\,dx = \ln x$ for $x > 0$. The answer is $\dfrac{1}{2}e^{2x} + \ln x + C$.

Answer to Question 2

a. The numerator is the derivative of the denominator, so let $u = x^2 + 1$, with $du = 2x\,dx$. The integral becomes $\int \dfrac{du}{u} = \ln|u| + C$. Since $x^2 + 1 > 0$, the answer is $\ln(x^2 + 1) + C$.

b. Let $u = x^2$, so $du = 2x\,dx$ and $x\,dx = \dfrac{1}{2}du$. Change the limits: $x = 0$ gives $u = 0$ and $x = 1$ gives $u = 1$. Then $\dfrac{1}{2}\int_0^1 e^u\,du = \dfrac{1}{2}[e^u]_0^1 = \dfrac{1}{2}(e - 1) = \dfrac{e - 1}{2}$, about 0.859.

c. Let $u = \sin x$, so $du = \cos x\,dx$. The integral becomes $\int u^3 du = \dfrac{u^4}{4} + C$. Substituting back gives $\dfrac{\sin^4 x}{4} + C$. Differentiating with the chain rule returns $\sin^3 x \cos x$.

Answer to Question 3

a. Take $u = x$, $v' = e^x$. Then $xe^x - \int e^x dx = (x-1)e^x + C$.

b. Take $u = \ln x$, $v' = x^2$, $v = \frac{x^3}{3}$: $\frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C = \frac{x^3}{9}(3\ln x - 1) + C$.

c. Take $u = x$, $v' = \sin x$, $v = -\cos x$: $-x\cos x + \int \cos x\, dx = \sin x - x\cos x + C$.

d. Write $\ln x = 1 \cdot \ln x$ with $u = \ln x$, $v' = 1$: $x\ln x - \int 1\, dx = x\ln x - x + C$.

Answer to Question 4

a. Factorise $x^2 - 1 = (x-1)(x+1)$ and write $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$. Multiplying out, $1 = A(x+1) + B(x-1)$. Putting $x = 1$ gives $A = \frac{1}{2}$; putting $x = -1$ gives $B = -\frac{1}{2}$. Integrating each term, $\frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + C = \frac{1}{2}\ln\frac{x-1}{x+1} + C$, using the laws of logarithms.

b. Write $\frac{3x+5}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$, so $3x + 5 = A(x+2) + B(x+1)$. At $x = -1$: $2 = A$. At $x = -2$: $-1 = -B$, so $B = 1$. Integrating, the answer is $2\ln(x+1) + \ln(x+2) + C$, which may also be written $\ln\left[(x+1)^2(x+2)\right] + C$.

Answer to Question 5

a. There is no direct antiderivative of $\sin^2 x$, so use the double-angle identity $\sin^2 x = \frac{1 - \cos 2x}{2}$. Integrating each term, the answer is $\frac{x}{2} - \frac{\sin 2x}{4} + C$.

b. Odd powers of sine are handled by keeping one factor: $\sin^3 x = \sin x(1 - \cos^2 x) = \sin x - \sin x \cos^2 x$. With $u = \cos x$, $\int \sin x \cos^2 x\, dx = -\frac{\cos^3 x}{3}$, so an antiderivative is $-\cos x + \frac{\cos^3 x}{3}$. Then $\left[-\cos x + \frac{\cos^3 x}{3}\right]_0^{\pi} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$.

c. Use the standard result $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$ with $a = 2$. The answer is $\frac{1}{2}\arctan\frac{x}{2} + C$.

Answer to Question 6

a. Let $F(u) = \int_0^u e^{t^2} dt$. By the fundamental theorem, $F'(u) = e^{u^2}$. The function we differentiate is $F(x^2)$, so by the chain rule $\frac{d}{dx} F(x^2) = F'(x^2) \cdot 2x = 2xe^{x^4}$.

b. The variable is in the lower limit, so swap the limits: $\int_x^3 \sin(t^2)\, dt = -\int_3^x \sin(t^2)\, dt$. By the fundamental theorem the derivative is $-\sin(x^2)$.

c. With 4 strips the width is $\frac{1}{4}$, and the left endpoints are 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. The sum is $\frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{1}{4} \cdot \frac{14}{16} = \frac{7}{32} =$ **0.21875**. The exact value is $\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \approx 0.3333$. The left sum is an underestimate because x^2 is increasing on $[0, 1]$, so each rectangle lies below the curve.

Answer to Question 7

a. $\int_0^4 (t^2 - 4t + 3)dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \frac{64}{3} - 32 + 12 = \frac{4}{3}$ **m**.

b. $t^2 - 4t + 3 = (t-1)(t-3) = 0$ at $t = 1$ and $t = 3$.

c. Let $F(t) = \frac{t^3}{3} - 2t^2 + 3t$: $F(0) = 0$, $F(1) = \frac{4}{3}$, $F(3) = 0$, $F(4) = \frac{4}{3}$. The velocity is positive on $(0, 1)$, negative on $(1, 3)$, positive on $(3, 4)$. Distance **4 m**.

d. Displacement counts backward motion as negative and so cancels it; distance adds up all the movement regardless of direction.

Answer to Question 8

a. The volume is the accumulated inflow: $V(t) = \int_0^t 12e^{-0.1s} ds$. An antiderivative of $12e^{-0.1s}$ is $-120e^{-0.1s}$, so $V(t) = \left[-120e^{-0.1s}\right]_0^t = 120\left(1 - e^{-0.1t}\right)$ litres.

b. Substituting $t = 30$: $V(30) = 120(1 - e^{-3}) \approx$ **114.0 litres**.

c. Solve $120(1 - e^{-0.1t}) = 100$: $e^{-0.1t} = \frac{1}{6}$, so $-0.1t = -\ln 6$ and $t = 10 \ln 6 \approx$ **17.9 minutes**.

d. As $t \to \infty$, $e^{-0.1t} \to 0$, so $V \to$ **120 litres**. This is the improper integral $\int_0^\infty 12e^{-0.1t} dt = 120$: the inflow slows down so fast that the total stays finite.

Answer to Question 9

a. $\frac{1}{24}\int_0^{24} \left(12 - 6\cos\frac{\pi t}{12}\right) dt = \frac{1}{24}\left[12t - \frac{72}{\pi}\sin\frac{\pi t}{12}\right]_0^{24} = \frac{288}{24} =$ **12 °C** (the sine term vanishes over a full period).

b. $\dfrac{1}{12}\left[12t - \dfrac{72}{\pi}\sin\dfrac{\pi t}{12}\right]_6^{18} = \dfrac{1}{12}\left(144 - \dfrac{72}{\pi}(-1-1)\right) = 12 + \dfrac{12}{\pi} \approx$ **15.8 °C**.

c. Between 06:00 and 18:00 the cosine term is mostly negative (the temperature is above 12 around noon), so we exclude the cold night hours.

d. $\cos\dfrac{\pi t}{12} = 0$ gives $\dfrac{\pi t}{12} = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, so **06:00 and 18:00**.

Answer to Question 10

a. $x = x^2$ gives $x = 0$ or $x = 1$: the points $(0, 0)$ and $(1, 1)$.

b. Area $= \int_0^1 (x - x^2)dx = \left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{6}$ square units. One square unit is $10\,000$ m^2, so the area is $\dfrac{10\,000}{6} \approx$ **1 667 m^2**.

c. $1667 \times 0.40 = 666.7$, so about **£670** to the nearest £10.

d. The road side is the diagonal of a unit square: $\sqrt{2} \times 100 \approx$ **141 m**.

Answer to Question 11

a. Cut the vase into thin slices perpendicular to the x-axis. Each slice of thickness dx is a disc of radius $y = \sqrt{x}$, so its volume is $\pi y^2 dx = \pi x\,dx$. Adding the slices, $V = \pi\int_0^{16} x\,dx = \pi\left[\dfrac{x^2}{2}\right]_0^{16} = 128\pi \approx$ **402 cm^3**.

b. Since 1 cm^3 = 1 ml, the volume is about **402 ml**.

c. Mass = density $\times$ volume $= 402.1 \times 1.6 \approx$ **643 g** of sand.

d. The open end is at $x = 16$, where the radius is $y = \sqrt{16} =$ **4 cm**, so the opening has diameter 8 cm.

Answer to Question 12

a. $\int_0^{0.3} 40x\,dx = 20(0.09) =$ **1.8 J**.

b. $\int_{0.3}^{0.5} 40x\,dx = 20(0.25 - 0.09) =$ **3.2 J**.

c. The force grows with extension, so each extra centimetre costs more energy. The second stretch is shorter (0.2 m) but happens at larger forces (12 N to 20 N).

d. $20x^2 = 10$ gives $x^2 = 0.5$, so $x =$ **0.707 m**.

Answer to Question 13

a. Using the given result, $\int_0^\infty f(x)\,dx = \frac{1}{4}\int_0^\infty xe^{-x/2}dx = \frac{1}{4}\times 4 = 1$. Since also $f(x)\geq 0$, f is a valid probability density.
b. Integrate by parts with $u = x$ and $v' = e^{-x/2}$, so $v = -2e^{-x/2}$: $\int_0^2 xe^{-x/2}dx = \left[-2xe^{-x/2}\right]_0^2 + 2\int_0^2 e^{-x/2}dx = -4e^{-1} + 2\left[-2e^{-x/2}\right]_0^2 = -4e^{-1} - 4e^{-1} + 4 = 4 - 8e^{-1}$. Dividing by 4: $P(X<2) = 1 - \frac{2}{e} \approx$ **0.264**.
c. The mean is $\frac{1}{4}\int_0^\infty x^2e^{-x/2}dx$. Using $\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$ with $n=2$, $a=\frac{1}{2}$, this is $\frac{1}{4}\cdot\frac{2}{(1/2)^3} = \frac{1}{4}\cdot 16 =$ **4 minutes**.
d. $P(X<2)\approx 0.264$, so $P(X>2)\approx$ **0.736**. She is much more likely to wait more than 2 minutes.

Answer to Question 14

a. The amount released is the integral of the rate: $\int_0^5 50e^{-0.2t}dt = \left[-250e^{-0.2t}\right]_0^5 = 250(1-e^{-1}) \approx$ **158 mg**.
b. The total is an improper integral: $\int_0^\infty 50e^{-0.2t}dt = \lim_{b\to\infty} 250(1-e^{-0.2b}) =$ **250 mg**, because $e^{-0.2b}\to 0$.
c. The fraction released in 5 hours is $\frac{250(1-e^{-1})}{250} = 1-e^{-1}\approx$ **63.2%** of the total.
d. Solve $250(1-e^{-0.2t}) = 0.9\times 250$: $1-e^{-0.2t} = 0.9$, so $e^{-0.2t} = 0.1$ and $t = 5\ln 10 \approx$ **11.5 hours**.

Answer to Question 15

a. Trapezium: $\frac{h}{2}[f_0 + 2(f_1+\cdots+f_5) + f_6] = 1\times[0 + 2(0.6+1.1+1.5+1.2+0.7)+0] =$ **10.2 m^2**.
b. Simpson: $\frac{2}{3}[0+4(0.6)+2(1.1)+4(1.5)+2(1.2)+4(0.7)+0] = \frac{2}{3}[2.4+2.2+6+2.4+2.8] = \frac{2}{3}(15.8)\approx$ **10.53 m^2**.
c. Flow = area × speed: trapezium $10.2\times 0.8 =$ **8.16 m^3/s**; Simpson $10.53\times 0.8\approx$ **8.43 m^3/s**.
d. Simpson's rule fits parabolas through triples of points and is usually more accurate for smooth shapes like a river bed, while the trapezium rule underestimates where the bed is curved outwards (concave down).

Answer to Question 16

a. On the left, $0.1P\left(1-\dfrac{P}{100}\right)=\dfrac{P(100-P)}{1000}$, so the reciprocal is $\dfrac{1000}{P(100-P)}$. On the right, $\dfrac{1}{P}+\dfrac{1}{100-P}=\dfrac{(100-P)+P}{P(100-P)}=\dfrac{100}{P(100-P)}$; multiplying by 10 gives the same expression.

b. Integrating term by term, $\int 10\left(\dfrac{1}{P}+\dfrac{1}{100-P}\right)dP=10\left(\ln P-\ln(100-P)\right)=10\ln\dfrac{P}{100-P}$ for $0<P<100$. Evaluating between the limits gives the result.

c. $T=10\left[\ln\dfrac{50}{50}-\ln\dfrac{20}{80}\right]=10\left[0-\ln\dfrac{1}{4}\right]=10\ln 4\approx$ **13.9 years**.

d. As $P_2\to 100$, $\dfrac{P_2}{100-P_2}\to\infty$, so its logarithm tends to infinity. The integral diverges: the population approaches the carrying capacity 100 but never reaches it in finite time.

Answer to Question 17

a. Discs of radius $\dfrac{1}{x}$ give $\pi y^2dx=\dfrac{\pi}{x^2}dx$. Then $V=\pi\left[-\dfrac{1}{x}\right]_1^{\infty}=\pi(0+1)=\pi$.

b. Since $\sqrt{1+x^{-4}}\geq 1$, the integrand is at least $\dfrac{1}{x}$.

c. $\int_1^b\dfrac{dx}{x}=\ln b\to\infty$ as $b\to\infty$. So the lower bound diverges, and $S=\infty$.

d. A finite volume can be filled with a finite amount of paint (π cubic units), but the horn has an infinite surface. The paint inside covers the infinite inside surface in a layer thinner and thinner as you go along. A real paint layer cannot become arbitrarily thin, so the paradox is a feature of the idealised mathematics.

Answer to Question 18

a. $I_0=\left[-e^{-t}\right]_0^{\infty}=0+1=$ **1**.

b. Take $u=t^n$, $v'=e^{-t}$: $I_n=\left[-t^ne^{-t}\right]_0^{\infty}+n\int_0^{\infty}t^{n-1}e^{-t}dt=0+nI_{n-1}$.

c. $I_1=1\cdot 1=$ **1**, $I_2=2\cdot 1=$ **2**, $I_3=3\cdot 2=$ **6**, $I_4=4\cdot 6=$ **24**. These are factorials, so $I_n=n!$.

d. The boundary term $\left[-t^ne^{-t}\right]_0^{\infty}$ vanishes at ∞ because exponentials grow faster than any power (l'Hôpital's rule applied n times), and at 0 because $t^n=0$ for $n\geq 1$.

Answer to Question 19

a. A disc at height y has radius $x = \sqrt{y}$, so area πy. Then $V = \pi \int_0^h y\,dy = \dfrac{\pi h^2}{2}$.

b. $V(9) = \dfrac{81\pi}{2} \approx$ **127 ml**.

c. $127.2 \div 5 \approx$ **25.4 seconds**.

d. Half the capacity is $\dfrac{81\pi}{4} = \dfrac{\pi}{2}h^2$, so $h^2 = \dfrac{81}{2}$ and $h = \dfrac{9}{\sqrt{2}} \approx$ **6.36 cm**. From $\dfrac{dV}{dt} = \pi h \dfrac{dh}{dt} = 5$ we get $\dfrac{dh}{dt} = \dfrac{5}{\pi h} \approx$ **0.250 cm/s**.

Answer to Question 20

a. Since f is continuous, the fundamental theorem says that $\dfrac{d}{dx}\int_0^x f(t)\,dt = f(x)$. Differentiating both sides of the equation, $f'(x) = 0 + f(x) = f(x)$. Setting $x = 0$, the integral is zero, so $f(0) = 1$.

b. Separating, $\dfrac{f'}{f} = 1$, so $\ln f = x + c$ and $f = Ae^x$. The condition $f(0) = 1$ gives $A = 1$, so $f(x) = e^x$.

c. Substituting back: $1 + \int_0^x e^t dt = 1 + (e^x - 1) = e^x$, which equals $f(x)$. The answer checks.

d. $f_1 = 1 + \int_0^x 1\,dt = 1 + x$, then $f_2 = 1 + x + \dfrac{x^2}{2}$ and $f_3 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$. Each step adds the next term of the Maclaurin series of e^x, so $f_n \to e^x$, in agreement with part (b).