Test answers - Functions, limits and continuity

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Substitution gives $\frac{0}{0}$, so factorise the numerator: $\frac{(x-3)(x+3)}{x-3} = x+3$ for $x \neq 3$ (1). The limit is therefore $3+3 = \mathbf{6}$ (1).

b. Again we get $\frac{0}{0}$. Rewrite using the standard limit $\frac{\sin u}{u} \to 1$: $\frac{\sin 2x}{\sin 3x} = \frac{2}{3} \cdot \frac{\sin 2x}{2x} \cdot \frac{3x}{\sin 3x}$ (1). As $x \to 0$ both fractions tend to 1, so the limit is $\frac{2}{3}$ (1).

In (a), cancelling $x-3$ is valid because a limit never uses the value at $x=3$ itself. In (b), a quick check with $x = 0.01$ gives $\sin 0.02 \div \sin 0.03 \approx 0.6667$.

Answer to Question 2 (4 marks)

Rough work: $|(2x+1)-9| = |2x-8| = 2|x-4|$, so it is enough to make $2|x-4| < \varepsilon$ (1). Proof: let $\varepsilon > 0$ be given and choose $\delta = \frac{\varepsilon}{2} > 0$ (1). Suppose $0 < |x-4| < \delta$ (1). Then $|(2x+1)-9| = 2|x-4| < 2\delta = \varepsilon$ (1). Since this works for every $\varepsilon > 0$, the definition is satisfied and $\lim_{x \to 4}(2x+1) = 9$.

The order matters: δ is chosen after ε and may depend on it. Any smaller positive δ, such as $\frac{\varepsilon}{3}$, also works. A common error is to assume what must be proved, starting from $|(2x+1)-9| < \varepsilon$.

Answer to Question 3 (4 marks)

a. The left-hand limit is $\lim_{x \to 1^-}(x^2 + a) = 1 + a$ (1). The right-hand value is $f(1) = 3 - 2 = 1$ (1). Continuity requires these to be equal, so $1 + a = 1$ and $a = 0$ (1).

b. With $a = 0$, the left piece x^2 has gradient $2x = 2$ at $x = 1$, while the right piece $3x - 2$ has gradient 3 (1). The one-sided derivatives differ, so f is not differentiable at 1: the graph has a corner there.

Answer to Question 4 (4 marks)

a. f is a polynomial, so it is continuous on $[1, 2]$. We have $f(1) = 1 + 2 - 6 = -3 < 0$ and $f(2) = 8 + 4 - 6 = 6 > 0$ (1). Since f changes sign, the intermediate value theorem gives a root in $(1, 2)$ (1).

b. Evaluate at the midpoint: $f(1.5) = 3.375 + 3 - 6 = 0.375 > 0$ (1). The sign change is now between 1 and 1.5, so the root lies in $(1, 1.5)$ (1). The interval has been halved; repeating the step would locate the root more precisely.

Answer to Question 5 (4 marks)

For every $x \neq 0$, $-1 \leq \cos \frac{1}{x} \leq 1$. Multiplying by $x^2 \geq 0$ keeps the inequalities, so $-x^2 \leq x^2 \cos \frac{1}{x} \leq x^2$ (2). Both bounds tend to 0 as $x \to 0$, so by the squeeze theorem the limit is 0 (1). On the other hand, $\cos \frac{1}{x}$ takes the value 1 at $x = \frac{1}{2n\pi}$ and -1 at $x = \frac{1}{(2n+1)\pi}$; both sequences tend to 0 but give different values, so no limit exists (1).