Lesson - Functions, limits and continuity

Year: University Year 1

Analysis begins with functions, limits and continuity. At university the definitions are precise and the proofs rigorous, using epsilon and delta.

Starter activity

Plot the values of $f(x) = \dfrac{\sin x}{x}$ on a calculator for $x = 1, 0.5, 0.1, 0.01$. What number do the outputs approach? The function is not defined at $x = 0$. Does it make sense to give it a value there? Now try $g(x) = \sin \dfrac{1}{x}$ with $x = 0.1, 0.01, 0.001$ and compare the behaviour.

Functions revisited

Definition: function, domain, codomain, image

A function $f : A \to B$ has a **domain** A, a **codomain** B and an **image** $f(A) = \{f(x) : x \in A\} \subseteq B$. The domain matters: $f(x) = \ln(x^2 - 1)$ has natural domain $\{x : |x| > 1\}$ and image $\mathbb{R}$.

Definition: monotonic

f is **increasing** if x, and *strictly* increasing if the inequality is strict. Strictly monotonic functions are injective.

Limits

Definition: epsilon-delta limit

$\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that 0. The value $f(a)$, or whether it exists, is irrelevant.

Rule: algebra of limits

If $\lim f = L$ and $\lim g = M$ then $\lim(f \pm g) = L \pm M$, $\lim fg = LM$, and $\lim f/g = L/M$ when $M \neq 0$.

Method: an epsilon-delta proof

1. State what you will prove and the definition used.

2. On scratch paper start from $|f(x) - L|$ and simplify to a multiple of $|x - a|$.
3. If another factor appears, bound it by restricting to $|x - a|$.
4. Choose δ as a formula in ε, for example $\delta = \min(1, \varepsilon/3)$.
5. Present the proof forwards, ending with "as required".

Example 1

Prove $\lim_{x \to 2}(3x - 1) = 5$**.**

Given $\varepsilon > 0$ choose $\delta = \varepsilon/3$. If 0 then $|(3x - 1) - 5| = 3|x - 2|$.

Example 2

Prove $\lim_{x \to 1} x^2 = 1$**.**

$|x^2 - 1| = |x - 1||x + 1|$. Restrict to $|x - 1|$, so 0 and $|x + 1|$. Then $|x^2 - 1|$. Choose $\delta = \min(1, \varepsilon/3)$; then $|x^2 - 1|$.

One-sided limits and limits at infinity

Property: one-sided limits

The **left limit** $\lim_{x \to a^-}$ and **right limit** $\lim_{x \to a^+}$ consider x and $x > a$ separately. The limit exists if and only if both exist and are equal. For example $|x|/x$ has left limit -1 and right limit 1 at 0, so no limit.

Rule: limit at infinity

$\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there is M with $x > M \Rightarrow |f(x) - L|$. For rational functions divide by the highest power: $\lim_{x \to \infty} \frac{3x^2 + x}{2x^2 - 5} = \frac{3}{2}$.

The squeeze theorem and standard limits

Theorem: squeeze theorem

If $g(x) \leq f(x) \leq h(x)$ near a and $\lim g = \lim h = L$, then $\lim f = L$.

Property: standard limits

$\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$, $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Example 3

Find $\lim_{x \to 0} x^2 \sin(1/x)$**.**

$-x^2 \le x^2 \sin(1/x) \le x^2$, and both bounds tend to 0, so by the squeeze theorem the limit is 0.

Continuity

Definition: continuity

f is **continuous at** a if $\lim_{x \to a} f(x) = f(a)$: the limit exists, $f(a)$ is defined and they agree. Sums, products, quotients (non-zero denominator) and compositions of continuous functions are continuous.

Property: types of discontinuity

- *Removable*: the limit exists but differs from $f(a)$.
- *Jump*: the one-sided limits differ.
- *Essential*: such as $\sin(1/x)$ at 0.

Example 4: removable discontinuity

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$ for $x \neq 3$. Is the discontinuity at 3 removable?

Factorising gives $f(x) = x + 3$ for $x \neq 3$, so the limit at 3 is 6. Defining $f(3) = 6$ makes the function continuous, so the discontinuity is removable. Simplify first; never substitute $x = 3$ into the original fraction.

The intermediate value theorem

Theorem: intermediate value theorem

If f is continuous on $[a, b]$ and k lies between $f(a)$ and $f(b)$, then $f(c) = k$ for some $c \in [a, b]$. In particular, if $f(a)$ and $f(b)$ have opposite signs then f has a root in (a, b).

Method: bisection

1. Find a, b with $f(a)$ and $f(b)$ of opposite signs.
2. Evaluate f at the midpoint.
3. Keep the half-interval where the sign changes.
4. Repeat. After k steps the root is within $(b - a)/2^k$.

Example 5

Locate the root of $f(x) = x^3 - 2x - 5$.

$f(2) = -1$ and $f(3) = 16$. The midpoint 2.5 gives $f(2.5) = 5.625 > 0$, so the root is in $(2, 2.5)$. Next $f(2.25) \approx 1.89 > 0$, so it is in $(2, 2.25)$. Ten steps give the root to within about 0.001.

Watch out

- Using $f(a)$ instead of the limit.
- Forgetting the restriction $|x - a|$ when bounding a factor.
- Applying the IVT without continuity on the whole closed interval: $1/x$ changes sign on $[-1, 1]$ with no root.

Proving things about limits

Method: showing that a limit does not exist

1. Look for two different ways of approaching the point, such as from the left and from the right, or along two sequences.
2. Compute the limit along each approach.
3. If the two answers are different, or if one is infinite, the limit does not exist.
4. State clearly which approaches you used, for example "along $x_n = \frac{1}{n\pi}$ the values are 0, but along $x_n = \frac{2}{(4n+1)\pi}$ they are 1".

Worked example 6: A limit at infinity with square roots

Evaluate $\lim_{x \to \infty} \left(\sqrt{x^2 + 3x} - x \right)$.

The form is $\infty - \infty$, so multiply and divide by the conjugate: $\dfrac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \dfrac{3x}{\sqrt{x^2 + 3x} + x}$. Divide numerator and denominator by x (which is positive): $\dfrac{3}{\sqrt{1 + 3/x} + 1}$. As $x \to \infty$ this tends to $\dfrac{3}{2}$. A common mistake is to say the limit is 0 because "$\sqrt{x^2} = x$"; the correction term $3x$ is not negligible.

Worked example 7: An epsilon-delta proof with a quotient

Prove that $\lim_{x \to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

Rough work: [redacted]. To keep $|x|$ away from zero, restrict to $|x - 2|$, so 1 and $\dfrac{1}{2|x|}$. Proof: given $\varepsilon > 0$ let $\delta = \min(1, 2\varepsilon)$. If 0, then [redacted]. As required. The idea is always the same: bound the awkward factor first, then choose δ as a minimum.

Worked example 8: The squeeze theorem with a trigonometric term

Evaluate $\lim_{x \to \infty} \dfrac{\sin x}{x}$.

Because $-1 \le \sin x \le 1$ and $x > 0$ we have $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$. Both outer functions tend to 0 as $x \to \infty$, so the squeeze theorem gives a limit of 0. Notice that $\sin x$ itself has no limit, but dividing by a quantity that grows without bound tames the oscillation completely.

Worked example 9: Choosing a constant for continuity

Find all values of a and b so that $f(x) = \{ax + 1 \quad x0$ is continuous at 0.

The left limit is $a \cdot 0 + 1 = 1$. The right limit is $\dfrac{\sin ax}{x} = a \cdot \dfrac{\sin ax}{ax} \to a$. Continuity needs left limit $=$ right limit $= f(0)$, so $a = 1$ and $b = 1$. Any other choice of a leaves a jump, and any other choice of b leaves a removable discontinuity.

Worked example 10: A fixed point of a continuous function

Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point, that is a number c with $f(c) = c$.

Let $g(x) = f(x) - x$, which is continuous on $[0,1]$. Then $g(0) = f(0) \ge 0$ because $f(0) \in [0,1]$, and $g(1) = f(1) - 1 \le 0$ because $f(1) \le 1$. If either value is zero, we have found the fixed point. Otherwise $g(0) > 0 > g(1)$, and the intermediate value theorem gives $c \in (0,1)$ with $g(c) = 0$, that is $f(c) = c$. Geometrically the graph of f must cross the diagonal.

Worked example 11: Bisection to a stated accuracy

How many bisection steps are needed to locate a root of a continuous function in $[2,3]$ to within 10^{-4}?

After k steps the interval has length $\dfrac{1}{2^k}$, so the midpoint is within $\dfrac{1}{2^{k+1}}$ of the root. We need $\dfrac{1}{2^{k+1}}$, that is $2^{k+1} > 10\,000$. Since $2^{13} = 8192$ and $2^{14} = 16\,384$, we need $k + 1 = 14$, so $k = 13$ steps. Bisection is slow but certain: each step gains exactly one binary digit.

Worked example 12: Limits of a piecewise function at a boundary

Let $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ for $x \ne 3$. Find $\lim_{x \to 3} f(x)$, decide whether f can be made continuous at 3, and state the natural domain.

Factorise the numerator: $x^2 - x - 6 = (x-3)(x+2)$. For $x \neq 3$ the fraction simplifies to $x+2$, so the limit as $x \to 3$ is 5. If we define $f(3) = 5$ the new function is continuous everywhere, so the break at 3 was removable. The natural domain of the original formula is every real number except 3, because the denominator vanishes there. It is the simplification, not the substitution, that reveals the true behaviour of the function near the point.

Worked example 13: Showing that a function is not continuous on an interval

Why does the intermediate value theorem not guarantee a root of $f(x) = \dfrac{x}{|x|}$ on $[-1, 1]$, even though $f(-1)$?

The function takes only the values -1 and 1 for $x \neq 0$, and is not defined at 0. It has a jump discontinuity at 0, so it is not continuous on the closed interval $[-1, 1]$. The theorem requires continuity at every point of the interval, and without it the graph can leap from below the axis to above it without ever crossing. This example shows that the hypotheses are not decoration: remove one and the conclusion fails.

Worked example 14: Using standard limits

Evaluate $\displaystyle\lim_{x \to 0} \frac{e^{3x} - 1}{x}$ and $\displaystyle\lim_{x \to 0} \frac{1 - \cos 4x}{x^2}$.

For the first limit write $\dfrac{e^{3x} - 1}{x} = 3 \cdot \dfrac{e^{3x} - 1}{3x}$. With $u = 3x \to 0$ the standard limit $\dfrac{e^u - 1}{u} \to 1$ gives the answer 3. For the second, $\dfrac{1 - \cos 4x}{x^2} = 16 \cdot \dfrac{1 - \cos u}{u^2}$ with $u = 4x$, and the standard limit $\dfrac{1}{2}$ gives $16 \times \dfrac{1}{2} = 8$. In both cases the trick is to make the argument of the function match the denominator.

Note: three kinds of limit statement

The notation $\lim_{x \to a} f(x) = L$ has three levels. Informally, $f(x)$ gets as close as we like to L when x is close to a. Formally, the ε-δ definition makes "as close as we like" and "close to" precise. In practice we use limit laws (sums, products, quotients), the squeeze theorem and standard limits. A proof from the definition is needed only when the question asks for it, but understanding the definition explains why the laws are true.

Watch out: common errors

- Substituting $x = a$ into a quotient that gives $\dfrac{0}{0}$ and writing "undefined": factorise or use a standard limit first.

- Choosing δ that depends on x. It may depend only on ε (and on a).
- Applying the intermediate value theorem to a function that is not continuous on the whole closed interval: $f(x) = \dfrac{1}{x}$ changes sign on $[-1, 1]$ with no root.
- Confusing "$f(a)$ exists" with "the limit exists": the limit ignores the value at a itself.

Did you know?

The rigorous ε-δ definition of a limit was developed in the nineteenth century by Augustin-Louis Cauchy and, above all, Karl Weierstrass. Before that, mathematicians used the idea of "infinitely small" quantities, which led to paradoxes. The new definition rescued calculus from criticism, such as the objections of Bishop Berkeley in 1734, who called infinitesimals "the ghosts of departed quantities".

Otto's tip

Write a rough work-out of δ in terms of ε on the side, then present the proof forwards.