Worksheet - Functions, limits and continuity

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Evaluating limits *(Section A · Skills practice)*

Evaluate each limit, or explain why it does not exist. Say what technique you use.

a. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ [2 marks]

b. $\lim_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{x}$ [3 marks]

c. $\lim_{x \to \infty} \dfrac{3x^2 + x}{2x^2 - 5}$ [3 marks]

d. $\lim_{x \to 0} \dfrac{\sin 3x}{x}$ [2 marks]

Question 2

Question 2 – Epsilon-delta proofs *(Section A · Skills practice)*

Use the ε-δ definition of a limit to prove each statement. Do your rough work first and then write the proof forwards.

a. $\lim_{x \to 2}(3x - 1) = 5$ [3 marks]

b. $\lim_{x \to 1} x^2 = 1$ [5 marks]

Question 3

Question 3 – One-sided limits *(Section A · Skills practice)*

Find the left-hand and right-hand limits at the stated point, and say whether the two-sided limit exists.

a. $f(x) = \dfrac{|x|}{x}$ at $x = 0$ [3 marks]

b. $g(x) = \begin{cases} x^2 + 1 & x \le 2 \\ 7 - x & x > 2 \end{cases}$ at $x = 2$ [3 marks]

c. $k(x) = \dfrac{1}{x - 1}$ at $x = 1$ [3 marks]

Question 4

Question 4 – Continuity and unknown constants *(Section A · Skills practice)*

Find the constants that make each function continuous at the given point.

a. $f(x) = \begin{cases} ax + 3 & x \leq 2 \\ x^2 + 1 & x > 2 \end{cases}$ at $x = 2$ [3 marks]

b. $g(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$ and $g(1) = a$ [3 marks]

Question 5

Question 5 – The intermediate value theorem *(Section A · Skills practice)*

Use the intermediate value theorem to prove that each equation has a solution in the stated interval. Check that all the conditions hold.

a. $x^3 + x - 1 = 0$ in $(0, 1)$ [3 marks]
b. $\cos x = x$ in $[0, 1]$ [4 marks]

Question 6

Question 6 – Domain, image and injectivity *(Section A · Skills practice)*

Answer each part.

a. Find the natural domain and the image of $f(x) = \ln(x^2 - 1)$. [3 marks]
b. Show that $g(x) = x^3 + x$ is injective on $\mathbb{R}$. [3 marks]
c. Is $h(x) = x^2$ injective on $\mathbb{R}$? On $[0, \infty)$? [2 marks]

Question 7

Question 7 – An electricity tariff *(Section B · Real-life practice)*

An energy company charges for the first 200 kWh each month at 30p per kWh. Above 200 kWh, there is a fixed charge of £k plus 20p for every kWh used. The monthly bill in pounds is $C(u) = \begin{cases} 0.30u & 0 \leq u \leq 200 \\ k + 0.20u & u > 200 \end{cases}$, where u is the number of kWh. The company wants the bill to have no sudden jump.

a. Find the bill for 200 kWh using the first rule.
b. Find k so that C is continuous at $u = 200$.
c. Find the bill for 350 kWh.
d. Explain what a jump in the tariff would mean for a customer using 200 kWh and for one using 201 kWh.

Question 8

Question 8 – Postage prices *(Section B · Real-life practice)*

A post office charges for a letter of weight w grams according to the table.

Weight w	Price
0	£1.20
100	£1.85
250	£2.70

a. Find the left-hand and right-hand limits of the price function $P(w)$ at $w = 100$, and at $w = 250$.
b. Is P continuous at $w = 100$? Classify the discontinuity.
c. What is $P(100)$? Is P left-continuous or right-continuous at $w = 100$?
d. Sketch the graph in words and state the cost of sending a 250 g letter and a 251 g letter.

Question 9

Question 9 – A day's temperature *(Section B · Real-life practice)*

The temperature in a greenhouse, in °C, t hours after midnight, is modelled by $T(t) = 9 - 6\cos\left(\dfrac{\pi t}{12}\right)$ for $0 \le t \le 24$.

a. Show that $T(6) = 9$ and find $T(14)$ to 1 decimal place.
b. Use the intermediate value theorem to show that the temperature is exactly 12°C at some time between 06:00 and 14:00.
c. Evaluate $T(10)$ to locate the time more precisely.
d. Solve $T(t) = 12$ exactly and check against part (c).

Question 10

Question 10 – Sizing a storage crate *(Section B · Real-life practice)*

A crate has a square base of side x metres and height $(x + 1)$ metres. The volume must be 10 m^3, so $f(x) = x^3 + x^2 - 10 = 0$.

a. Show that $f(1) = -8$ and $f(2) = 2$ and explain why there is a solution in $(1, 2)$.
b. Carry out three steps of the bisection method starting from $[1, 2]$ and complete a table of midpoints and signs.
c. State the interval of length $\frac{1}{8}$ that contains the root.
d. After how many bisection steps starting from $[1, 2]$ is the root guaranteed to be within 0.001?

Question 11

Question 11 – A falling stone *(Section B · Real-life practice)*

A stone is dropped from a bridge. After t seconds it has fallen $s = 4.9t^2$ metres. The average speed between $t = 2$ and $t = 2 + h$ is $\dfrac{s(2+h) - s(2)}{h}$.

h (s)	1	0.1	0.01
Average speed (m/s)	?	?	?

a. Show that the average speed is $19.6 + 4.9h$.
b. Complete the table.
c. Find the limit as $h \to 0$ and explain what it means.
d. Find the speed after 3 seconds in the same way.

Question 12

Question 12 – Tolerance on a machine *(Section B · Real-life practice)*

A machine cuts rods whose length in cm is $L = 3d - 1$, where d is a dial setting. The setting $d = 2$ should give 5 cm. Rods are accepted if they are within ε cm of 5 cm.

a. Show that $|L - 5| = 3|d - 2|$.
b. How close to 2 must the dial be if $\varepsilon = 0.03$?
c. How close for $\varepsilon = 0.003$?
d. Explain how this is an example of the definition $\lim_{d \to 2}(3d - 1) = 5$.

Question 13

Question 13 – Long-term population *(Section B · Real-life practice)*

The number of birds in a colony after t years on a protected island is modelled by $P(t) = \dfrac{1200t}{t + 3}$.

a. Find $P(3)$ and $P(27)$.
b. Find $\lim_{t \to \infty} P(t)$ and interpret it.
c. Show that $1200 - P(t) = \dfrac{3600}{t + 3}$.
d. Use part (c) to find how large t must be for the population to be within 1 bird of the limit.

Question 14

Question 14 – A vibrating spring *(Section B · Real-life practice)*

The displacement in cm of a spring in a lab is $y(t) = e^{-0.5t}\cos 3t$ for $t \geq 0$ (in seconds).

a. Show that $-e^{-0.5t} \leq y(t) \leq e^{-0.5t}$.
b. Use the squeeze theorem to find $\lim_{t\to\infty} y(t)$.
c. After what time is $|y(t)|$ certain to be below 0.01 cm?
d. Does y have a limit if the exponential factor is removed, so that $y = \cos 3t$? Explain.

Question 15

Question 15 – Designing a track *(Section C · Problem solving)*

A railway engineer joins two pieces of track with a straight middle section. The height profile (in metres) as a function of x (in hundreds of metres) is $f(x) = \{2x \quad x3$.

a. Write the two equations that a and b must satisfy for f to be continuous at $x = 1$ and at $x = 3$.
b. Solve for a and b.
c. Work out the gradients of the three pieces. Is the track smooth (differentiable) at $x = 1$ and $x = 3$?
d. What would a train passenger feel at a point where the gradient changes suddenly? Suggest a better design.

Question 16

Question 16 – Classifying discontinuities in sensor data *(Section C · Problem solving)*

A data logger records four quantities, each modelled by a function with a break at the stated point. Classify each discontinuity as removable, jump, infinite, or essential (oscillating).

Function	Point
$f_1(x) = \dfrac{x^2 - 1}{x - 1}$	$x = 1$
$f_2(x) = \dfrac{\lvert x\rvert}{x}$	$x = 0$
$f_3(x) = \dfrac{1}{x^2}$	$x = 0$
$f_4(x) = \sin\dfrac{1}{x}$	$x = 0$

a. Classify each discontinuity and justify your answer with limits.
b. For the removable one, say how to redefine the function so that it is continuous.
c. Which of the four cannot be repaired by changing the value at the single point?

d. Sketch (in words) the behaviour of f_4 near 0, using the points $x = \dfrac{1}{n\pi}$.

Question 17

Question 17 – The hikers' problem *(Section C · Problem solving)*

On Saturday, Zara walks up a mountain path from the car park (height 0 m) at 08:00 to the summit (height 1 200 m) at 16:00, and her height at time t is $u(t) = 18.75(t-8)^2$. On Sunday she walks down the same path, starting at the summit at 08:00 and reaching the car park at 16:00 with height $d(t) = 1200 - 150(t-8)$.

a. Check $u(8)$, $u(16)$, $d(8)$ and $d(16)$.
b. Let $g(t) = u(t) - d(t)$. Show that $g(8)$ and explain why Zara is at the same height at the same time of day on both days at some moment.
c. Find that time by solving $g(t) = 0$.
d. Would the conclusion still hold if she rested on the way and her height were only piecewise continuous? Explain.

Question 18

Question 18 – Comparing growth of algorithms *(Section C · Problem solving)*

A computer scientist compares the number of steps of three algorithms with input size n: $A(n) = n^2 + 5n$, $B(n) = 3n^2 - 1$ and $C(n) = 2^n$.

a. Find $\displaystyle\lim_{n\to\infty} \frac{A(n)}{B(n)}$ and interpret it.
b. Find $\displaystyle\lim_{n\to\infty} \frac{n^2}{2^n}$ using l'Hôpital's rule twice (treat n as a real variable).
c. What does part (b) say about algorithm C compared with A?
d. Evaluate $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$ by multiplying by the conjugate.

Question 19

Question 19 – Cutting a square plate *(Section C · Problem solving)*

A square metal plate has side x cm and area x^2 cm^2. The ideal side is 2 cm, so the ideal area is 4 cm^2. The area must be within ε cm^2 of 4.

a. Show that $|x^2 - 4| = |x-2||x+2|$.
b. If $|x-2|$ show that $|x+2|$.
c. Prove, using $\delta = \min\left(1, \dfrac{\varepsilon}{5}\right)$, that $\displaystyle\lim_{x\to 2} x^2 = 4$.
d. Find a suitable δ when the area tolerance is $\varepsilon = 0.1$ cm^2, and say what it means for the workshop.

Question 20

Question 20 – Challenge: the chord theorem *(Section D · Challenge)*

Let $T(t)$ be the temperature at time t hours (a continuous function) with $T(0) = T(24)$, so the temperature at midnight is the same on two consecutive days. Prove that there are two times exactly 12 hours apart with the same temperature.

a. Define $g(t) = T(t + 12) - T(t)$ for $0 \leq t \leq 12$. Why is g continuous?
b. Show that $g(0) = -g(12)$.
c. Deduce that $g(c) = 0$ for some $c \in [0, 12]$.
d. Explain what this proves about the temperature, and why it works for any continuous function on a circle (such as the temperature around the equator at a given moment).