Answers - Functions, limits and continuity

Year: University Year 1

Answer to Question 1

a. Direct substitution gives $\frac{0}{0}$, so we factorise: $\frac{(x-2)(x+2)}{x-2} = x+2$ for $x \neq 2$. The limit does not depend on the value at 2, so it is $2+2 = \mathbf{4}$.

b. Again $\frac{0}{0}$. Multiply top and bottom by the conjugate $\sqrt{1+x}+1$: $\frac{(1+x)-1}{x(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+x}+1}$, which tends to $\frac{1}{2}$.

c. Divide numerator and denominator by the highest power x^2: $\frac{3+1/x}{2-5/x^2}$. As $x \to \infty$, $1/x \to 0$ and $5/x^2 \to 0$, so the limit is $\frac{3}{2}$.

d. Use the standard limit: $\frac{\sin 3x}{x} = 3 \cdot \frac{\sin 3x}{3x}$. With $u = 3x \to 0$, $\frac{\sin u}{u} \to 1$, so the limit is **3**.

Answer to Question 2

a. Rough work: $|(3x-1)-5| = |3x-6| = 3|x-2|$, so we want $3|x-2| < \varepsilon$, which suggests $\delta = \varepsilon/3$. Proof: let $\varepsilon > 0$ be given and choose $\delta = \frac{\varepsilon}{3} > 0$. If $0 < |x-2| < \delta$, then $|(3x-1)-5| = 3|x-2| < 3\delta = \varepsilon$. Since ε was arbitrary, $\lim_{x \to 2}(3x-1) = 5$.

b. Rough work: $|x^2-1| = |x-1|\,|x+1|$. We must control $|x+1|$: if $|x-1| < 1$ then $0 < x < 2$, so $|x+1| < 3$. Proof: let $\varepsilon > 0$ and choose $\delta = \min\left(1, \frac{\varepsilon}{3}\right)$. If $0 < |x-1| < \delta$, then $|x+1| < 3$ because $\delta \leq 1$, and $|x-1| < \varepsilon/3$ because $\delta \leq \varepsilon/3$. Hence $|x^2-1| < \frac{\varepsilon}{3} \cdot 3 = \varepsilon$, as required.

Answer to Question 3

a. For x, $|x| = -x$, so $f = -1$ and the left limit is -1. For $x > 0$, $f = 1$ and the right limit is 1. The limits differ, so **no limit exists**.

b. Left limit: $2^2 + 1 = 5$. Right limit: $7 - 2 = 5$. They agree, so the limit is **5**. (Also $g(2) = 5$, so g is continuous at 2.)

c. As $x \to 1^+$, $k \to +\infty$; as $x \to 1^-$, $k \to -\infty$. **The limit does not exist** (neither side is finite).

Answer to Question 4

a. For continuity at 2, the value $f(2)$ must equal the limits from both sides. From the left (and at $x = 2$), $f(2) = 2a + 3$. From the right, $\lim_{x \to 2^+} (x^2 + 1) = 5$. Setting $2a + 3 = 5$ gives $a = 1$, and then both pieces join at the point $(2, 5)$.

b. For $x \neq 1$, factorise: $g(x) = \dfrac{(x-1)(x+1)}{x-1} = x + 1$, so $\lim_{x \to 1} g(x) = 2$. Continuity at 1 requires $g(1) = \lim_{x \to 1} g(x)$, so $a = 2$. This fills the hole in the graph of $y = x + 1$.

Answer to Question 5

a. Let $f(x) = x^3 + x - 1$. It is a polynomial, so it is continuous on $[0, 1]$, the first condition of the IVT. Next, $f(0) = -1 < 0$ and $f(1) = 1 + 1 - 1 = 1 > 0$, so 0 lies between $f(0)$ and $f(1)$. By the IVT there is $c \in (0, 1)$ with $f(c) = 0$: the equation has a root in $(0, 1)$.

b. Rewrite the equation as $f(x) = \cos x - x = 0$. The function f is a difference of continuous functions, so it is continuous on $[0, 1]$. Then $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. The signs differ, so by the IVT $f(c) = 0$ for some $c \in (0, 1)$, that is $\cos c = c$.

Answer to Question 6

a. The logarithm needs $x^2 - 1 > 0$, that is $|x| > 1$: domain $(-\infty, -1) \cup (1, \infty)$. As $|x|$ ranges over $(1, \infty)$, $x^2 - 1$ takes every positive value, so $\ln$ takes every real value: image $\mathbb{R}$.

b. $g'(x) = 3x^2 + 1 > 0$ for all x, so g is strictly increasing and therefore injective.

c. On $\mathbb{R}$ it is not injective, since $h(-2) = h(2)$. On $[0, \infty)$ it is strictly increasing, so it is **injective**.

Answer to Question 7

a. $0.30 \times 200 =$ **£60**.

b. Right-hand limit at 200: $k + 0.20 \times 200 = k + 40$. Continuity needs $k + 40 = 60$, so $k = 20$.

c. $C(350) = 20 + 0.20 \times 350 = 20 + 70 =$ **£90**.

d. With a jump, someone using 201 kWh could pay noticeably more (or less) than someone using exactly 200 kWh for a one kWh difference. Continuity removes this unfairness.

Answer to Question 8

a. At $w = 100$: left limit **£1.20**, right limit **£1.85**. At $w = 250$: left limit **£1.85**, right limit **£2.70**.
b. No, because the one-sided limits differ. This is a **jump discontinuity** of size £0.65 at 100 (and £0.85 at 250).
c. $P(100) =$ £1.20 equals the left limit, so P is **left-continuous** but not right-continuous at 100.
d. The graph is a staircase with open circles at the left end of each step and closed circles at the right end. A 250 g letter costs **£1.85** and a 251 g letter costs **£2.70**.

Answer to Question 9

a. $T(6) = 9 - 6\cos\dfrac{\pi}{2} = 9 - 0 = 9$. Also $T(14) = 9 - 6\cos\dfrac{7\pi}{6} = 9 + 6 \times 0.866 =$ **14.2 °C**.
b. T is built from a cosine, so it is continuous on $[6, 14]$. At 06:00 the temperature is $9 < 12$ and at 14:00 it is $14.2 > 12$. By the IVT, $T(t) = 12$ for some $t \in (6, 14)$.
c. $T(10) = 9 - 6\cos\dfrac{5\pi}{6} = 9 + 5.196 = 14.196 > 12$. Since $T(6) < 12 < T(10)$, applying the IVT again shows that the time lies in $(6, 10)$.
d. $T(t) = 12$ gives $\cos\dfrac{\pi t}{12} = -\dfrac{1}{2}$, so $\dfrac{\pi t}{12} = \dfrac{2\pi}{3}$ (the solution in the range) and $t =$ **8**, that is 08:00. This lies in $(6, 10)$, as predicted.

Answer to Question 10

a. $f(1) = 1 + 1 - 10 = -8$ and $f(2) = 8 + 4 - 10 = 2$. The polynomial is continuous, so the IVT gives a root in $(1, 2)$.
b. Midpoint 1.5: $f = -4.375$, root in $(1.5, 2)$. Midpoint 1.75: $f = -1.578$, root in $(1.75, 2)$. Midpoint 1.875: $f = 0.107 > 0$, root in $(1.75, 1.875)$.
c. The root lies in $(1.75, 1.875)$, an interval of length $0.125 = \frac{1}{8}$.
d. After k steps the interval has length $\dfrac{1}{2^k}$. We need $\dfrac{1}{2^k}$, so $2^k > 1000$, giving $k =$ **10** (since $2^{10} = 1024$).

Answer to Question 11

a. Expand: $\dfrac{4.9(2+h)^2 - 4.9 \times 4}{h} = \dfrac{4.9(4 + 4h + h^2 - 4)}{h} = \dfrac{4.9(4h + h^2)}{h} = 19.6 + 4.9h$ for $h \neq 0$.
b. Substituting into $19.6 + 4.9h$: for $h = 1$, **24.5**; for $h = 0.1$, **20.09**; for $h = 0.01$,

19.649 m/s.

c. As $h \to 0$ the average speed tends to **19.6 m/s**. This is the instantaneous speed at $t = 2$: the limit of average speeds over shorter and shorter intervals, that is the derivative $s'(2)$.

d. In the same way, $\dfrac{4.9(3+h)^2 - 4.9 \times 9}{h} = \dfrac{4.9(6h + h^2)}{h} = 29.4 + 4.9h \to$ **29.4 m/s** after 3 seconds.

Answer to Question 12

a. $L - 5 = 3d - 1 - 5 = 3d - 6 = 3(d - 2)$, so taking absolute values $|L - 5| = 3|d - 2|$.

b. We need $|L - 5| < 0.03$, that is $3|d - 2| < 0.03$, so $|d - 2| <$ **0.01**: the dial must be between 1.99 and 2.01.

c. Similarly $3|d - 2| < 0.003$ gives $|d - 2| <$ **0.001**. A tolerance ten times smaller needs a dial ten times more precise.

d. For every tolerance $\varepsilon > 0$ we can find a precision $\delta = \varepsilon/3$ such that $|d - 2| < \delta$ guarantees $|L - 5| < \varepsilon$. That is exactly the ε-δ definition of $\lim\limits_{d \to 2}(3d - 1) = 5$.

Answer to Question 13

a. $P(3) = \dfrac{3600}{6} =$ **600**; $P(27) = \dfrac{32400}{30} =$ **1 080**.

b. Divide by t: $P = \dfrac{1200}{1 + 3/t} \to$ **1 200**. The island can support about 1 200 birds in the long run.

c. $1200 - \dfrac{1200t}{t+3} = \dfrac{1200(t+3) - 1200t}{t+3} = \dfrac{3600}{t+3}$.

d. We need $\dfrac{3600}{t+3}$, so $t + 3 > 3600$ and $t >$ **3 597 years**. This is the "M" in the definition of a limit at infinity, for $\varepsilon = 1$.

Answer to Question 14

a. Since $-1 \le \cos 3t \le 1$ and $e^{-0.5t} > 0$, multiply through by $e^{-0.5t}$.

b. Both bounds tend to 0 as $t \to \infty$, so by the squeeze theorem $y(t) \to 0$.

c. It is enough that $e^{-0.5t}$, so $0.5t > \ln 100$ and $t > 2 \ln 100 \approx$ **9.21 seconds**.

d. No. $\cos 3t$ keeps oscillating between -1 and 1, so it has no limit as $t \to \infty$.

Answer to Question 15

a. At $x = 1$: $2(1) = a + b$, so $a + b = 2$. At $x = 3$: $3a + b = 3^2 - 1 = 8$.
b. Subtract: $2a = 6$, so $a = 3$, and then $b = 2 - 3 = -1$. So $a = 3$, $b = -1$.
c. The gradients are 2, 3 and $2x$ (which equals 6 at $x = 3$). At $x = 1$ the gradient jumps from 2 to 3, and at $x = 3$ from 3 to 6. So f is **continuous but not differentiable** at both points: there are corners.
d. A sudden change of gradient would feel like a jolt. Engineers use smooth transition curves (such as parabolas or clothoids) so that the gradient also changes continuously.

Answer to Question 16

a. f_1: $f_1 = x + 1$ for $x \neq 1$, so the limit is 2: **removable**. f_2: one-sided limits -1 and 1: **jump**. f_3: $f_3 \to +\infty$ from both sides: **infinite**. f_4: oscillates between -1 and 1: **essential**.
b. Define $f_1(1) = 2$.
c. f_2, f_3 and f_4 cannot be repaired: their limits do not exist (or are infinite).
d. At $x = \dfrac{1}{n\pi}$ the function equals $\sin(n\pi) = 0$, while at $x = \dfrac{2}{(2n+1)\pi}$ it equals ± 1. Both kinds of point get arbitrarily close to 0, so the values keep switching between -1 and 1 and no limit exists.

Answer to Question 17

a. $u(8) = 0$, $u(16) = 18.75 \times 64 = 1200$, $d(8) = 1200$, $d(16) = 0$.
b. $g(8) = 0 - 1200 = -1200$ and $g(16) = 1200 - 0 = 1200 > 0$. Both u and d are continuous, so g is continuous, and by the IVT $g(t) = 0$ for some $t \in (8, 16)$. At that moment $u(t) = d(t)$.
c. Let $s = t - 8$: $18.75s^2 = 1200 - 150s$, so $s^2 + 8s - 64 = 0$ and $s = -4 + \sqrt{80} \approx 4.944$. So $t \approx$ **12.94**, about **12:57**. The common height is about **458 m**.
d. If the path had jumps (not realistic for walking) the IVT would not apply. Zara's height never jumps, so continuity holds even when she rests (the graph is then flat, which is still continuous).

Answer to Question 18

a. Divide by n^2: $\dfrac{1 + 5/n}{3 - 1/n^2} \to \dfrac{1}{3}$. For large n, A takes about a third as many steps as B.
b. $\dfrac{n^2}{2^n} \to \dfrac{2n}{2^n \ln 2} \to \dfrac{2}{2^n (\ln 2)^2} \to$ **0**.
c. Since $n^2/2^n \to 0$, algorithm C takes far more steps than any quadratic algorithm

once n is large: exponential growth beats polynomial growth.

d. $\dfrac{(n^2+n)-n^2}{\sqrt{n^2+n}+n} = \dfrac{n}{\sqrt{n^2+n}+n} = \dfrac{1}{\sqrt{1+1/n}+1} \to \dfrac{1}{2}$.

Answer to Question 19

a. By the difference of two squares, $x^2-4=(x-2)(x+2)$. Taking absolute values and using $|ab|=|a||b|$, $|x^2-4|=|x-2||x+2|$.
b. $|x-2|<1$ means $1<x<3$. Adding 2, $3<x+2<5$, so $|x+2|<5$.
c. Given $\varepsilon>0$, let $\delta=\min(1,\varepsilon/5)$. If $0<|x-2|<\delta$, then $|x-2|<1$, so $|x+2|<5$ by part (b); and $|x-2|<\varepsilon/5$. Hence $|x^2-4|<\dfrac{\varepsilon}{5}\times 5=\varepsilon$, which proves $\lim\limits_{x\to 2} x^2=4$.
d. With $\varepsilon=0.1$, $\delta=\min(1,0.02)=$ **0.02 cm**. The workshop must cut the side to within 0.02 cm of 2 cm to guarantee the area is within 0.1 cm^2 of 4 cm^2.

Answer to Question 20

a. T is continuous, so $t\mapsto T(t+12)$ is continuous, and the difference of continuous functions is continuous.
b. $g(0)=T(12)-T(0)$ and $g(12)=T(24)-T(12)=T(0)-T(12)=-g(0)$.
c. If $g(0)=0$ we are done. Otherwise $g(0)$ and $g(12)$ have opposite signs, so by the IVT there is $c\in(0,12)$ with $g(c)=0$.
d. Then $T(c+12)=T(c)$: the temperature at time c equals the temperature 12 hours later. The same argument shows that on a circle (such as the equator) there are two opposite points with the same temperature.