Test answers - Eigenvalues and eigenvectors

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

The characteristic equation is $\det(A - \lambda I) = (1-\lambda)^2 - 4 = 0$ (1). So $1 - \lambda = \pm 2$, which gives $\lambda = 3$ or $\lambda = -1$ (1). For $\lambda = 3$: the system $-2x + 2y = 0$ gives $x = y$, so $\mathbf{v} = (1, 1)$ (1). For $\lambda = -1$: $2x + 2y = 0$ gives $y = -x$, so $\mathbf{v} = (1, -1)$ (1). Check: trace $2 = 3 + (-1)$ and determinant $-3 = 3 \times (-1)$.

Eigenvectors are only determined up to a non-zero multiple, so $(2, 2)$ or $(-1, 1)$ also score full marks. The matrix is symmetric, and its two eigenvectors are perpendicular: $(1, 1) \cdot (1, -1) = 0$.

Answer to Question 2 (4 marks)

The matrix is upper triangular, so its eigenvalues are the diagonal entries 3 and 2 (1). For $\lambda = 3$: $(A - 3I)\mathbf{v} = \mathbf{0}$ gives $y = 0$, so $\mathbf{v} = (1, 0)$; for $\lambda = 2$: $x + y = 0$, so $\mathbf{v} = (1, -1)$ (1). Hence $P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $D = \operatorname{diag}(3, 2)$, and here $P^{-1} = P$ (1). Therefore $A^n = PD^nP^{-1} = \begin{pmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{pmatrix}$ (1). For $n = 1$ this gives A back, as a check.

Check $P^{-1} = P$: $P^2 = I$. The diagonalisation works because the eigenvalues are distinct.

Answer to Question 3 (4 marks)

a. The characteristic equation is $(-\lambda)^2 - (-4)(1) = \lambda^2 + 4 = 0$, so $\lambda^2 = -4$ and $\lambda = \pm 2i$ (2). A real matrix can have complex eigenvalues; they come in conjugate pairs and here describe a rotation combined with a stretch.

b. The matrix is triangular, so $\lambda = 1$ is a repeated eigenvalue (1). Solving $(A - I)\mathbf{v} = \mathbf{0}$, the only condition is $y = 0$, so every eigenvector is a multiple of $(1, 0)$ (1). There is only one independent eigenvector, so the matrix cannot be diagonalised.

Answer to Question 4 (4 marks)

a. The trace is $0.8 + 0.9 = 1.7$ and the determinant is $0.72 - 0.02 = 0.70$ (1). The eigenvalues satisfy $\lambda^2 - 1.7\lambda + 0.7 = (\lambda - 1)(\lambda - 0.7) = 0$, so they are 1 and 0.7 (1). A transition matrix always has eigenvalue 1.

b. The long-term state is the eigenvector for $\lambda = 1$: $-0.2c + 0.1s = 0$, so $s = 2c$ (1).

Since $c + s = 1$, $3c = 1$ and the cinema share tends to $\frac{1}{3}$, about 33% (1). The other part decays like 0.7^n.

Check: $(\frac{1}{3}, \frac{2}{3})$ is unchanged by the matrix.

Answer to Question 5 (4 marks)

For a 2×2 matrix the characteristic equation is $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$. Here $\operatorname{tr} A = 1$ and $\det A = 0 - 6 = -6$, so $\lambda^2 - \lambda - 6 = 0$ (1). By Cayley-Hamilton, $A^2 - A - 6I = 0$, that is $A^2 = A + 6I$ (1). Factorising, $A(A - I) = 6I$ (1), so $A^{-1} = \frac{1}{6}(A - I) = \frac{1}{6}\begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$ (1). Check: multiplying by A gives I.

Check explicitly: $A^2 = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$, and $A + 6I = \begin{pmatrix} 7 & 2 \\ 3 & 6 \end{pmatrix}$. The usual formula for a 2×2 inverse, $\frac{1}{-6}\begin{pmatrix} 0 & -2 \\ -3 & 1 \end{pmatrix}$, gives the same matrix. Since $\det A \neq 0$, the inverse exists, consistent with 0 not being an eigenvalue.