Worksheet - Eigenvalues and eigenvectors

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Eigenvalues and eigenvectors *(Section A · Skills practice)*

Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

a. Write down and solve the characteristic equation $\det(A - \lambda I) = 0$. [3 marks]
b. Find an eigenvector for each eigenvalue. [4 marks]
c. Check that the trace equals the sum of the eigenvalues and the determinant equals their product. [2 marks]

Question 2

Question 2 – Triangular and 3 by 3 matrices *(Section A · Skills practice)*

Find the eigenvalues of each matrix.

a. $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ [2 marks]

b. [redacted] [4 marks]

c. A 2×2 matrix has trace 7 and determinant 10. [3 marks]

Question 3

Question 3 – Complex eigenvalues *(Section A · Skills practice)*

Find the eigenvalues of each real matrix and give a complex eigenvector for one of them.

a. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (a rotation through 90°) [3 marks]

b. $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ [4 marks]

Question 4

Question 4 – Diagonalisation *(Section A · Skills practice)*

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

a. Find the eigenvalues and eigenvectors of A. [4 marks]
b. Write $A = PDP^{-1}$ with D diagonal. [3 marks]
c. Use it to compute A^5. [3 marks]

Question 5

Question 5 – Defective matrices *(Section A · Skills practice)*

Let $N = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

a. Find the eigenvalue and its algebraic multiplicity. [2 marks]
b. Find all eigenvectors. What is the geometric multiplicity? [3 marks]
c. Explain why N cannot be diagonalised. [2 marks]

Question 6

Question 6 – Eigenvalues of related matrices *(Section A · Skills practice)*

Let A have eigenvalues 2 and 5 (for example $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$).

a. Find the eigenvalues of A^2, A^{-1} and $A + 3I$. [4 marks]
b. Find $\det(A^2)$ and $\operatorname{tr}(A^2)$ from the eigenvalues, and check with direct calculation. [4 marks]

Question 7

Question 7 – Switching phone networks *(Section B · Real-life practice)*

Each month 10% of customers of Network A switch to Network B, and 20% of customers of Network B switch to Network A. Let $\mathbf{v}_n = (a_n, b_n)^T$ be the proportions after n months, so $\mathbf{v}_{n+1} = P\mathbf{v}_n$ with $P = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$.

a. Show that 1 is an eigenvalue and find the second eigenvalue.
b. Find the eigenvectors.
c. Find the long-term market shares.
d. Starting from $\mathbf{v}_0 = (1, 0)^T$, find $\mathbf{v}_n$ and the share of Network A after 5 months.

Question 8

Question 8 – A population of rabbits in two age groups *(Section B · Real-life practice)*

A rabbit population has juveniles J and adults A. Each period every adult produces 2 juveniles, 50% of juveniles become adults, and 80% of adults survive. So $\begin{pmatrix} J' \\ A' \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0.5 & 0.8 \end{pmatrix} \begin{pmatrix} J \\ A \end{pmatrix}$.

a. Write the characteristic equation and find the eigenvalues to 3 decimal places.
b. Which eigenvalue governs the long-term growth? What is the growth rate per period?
c. Find the long-term ratio of adults to juveniles.
d. Explain why the other eigenvalue does not matter in the long run.

Question 9

Question 9 – Vibrating carriages *(Section B · Real-life practice)*

Two carriages of mass m are joined by springs of stiffness k. For displacements $\mathbf{x} = (x_1, x_2)$ the equations of motion are $\ddot{\mathbf{x}} = -\dfrac{k}{m} K\mathbf{x}$ where $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. A normal mode $\mathbf{x} = \mathbf{v}\cos\omega t$ requires $K\mathbf{v} = \mu\mathbf{v}$ with $\omega^2 = \dfrac{k\mu}{m}$.

a. Find the eigenvalues μ of K.
b. Find the eigenvectors and describe the two modes of motion.
c. Find the two angular frequencies if $k = 300$ N/m and $m = 3$ kg.
d. Which mode is faster? Explain physically.

Question 10

Question 10 – Stretching a photograph *(Section B · Real-life practice)*

A photo-editing program stretches an image by the matrix $S = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ applied to coordinates relative to the centre.

a. Find the eigenvalues of S.
b. Find the directions that are only stretched, not turned.
c. By what factor is each of these directions stretched?
d. Find the area scale factor and check that it equals the product of the eigenvalues.

Question 11

Question 11 – Cooling two rooms *(Section B · Real-life practice)*

Two connected rooms have temperature deviations $\mathbf{x} = (x_1, x_2)$ from the outside temperature (in °C). They satisfy $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ (per hour). Initially $\mathbf{x}(0) = (4, 0)$.

a. Find the eigenvalues and eigenvectors of A.
b. Write the general solution $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$.
c. Find the constants c_1, c_2 and the temperatures after 1 hour.
d. Explain why both rooms eventually reach the outside temperature.

Question 12

Question 12 – Ranking three web pages *(Section B · Real-life practice)*

A tiny website has three pages. Page A links to B and C, page B links to C, and page C links to A. The link matrix is $L =$ , where each column shows how a page shares its importance among the pages it links to. The ranking vector satisfies $L\mathbf{r} = \mathbf{r}$.

a. Check that each column of L sums to 1.
b. Solve $L\mathbf{r} = \mathbf{r}$ with $r_A + r_B + r_C = 1$.
c. Which pages rank highest?
d. Explain why page B ranks lower than C although both receive a link from A.

Question 13

Question 13 – Spirals in a sunflower *(Section B · Real-life practice)*

Spiral counts in sunflowers are often consecutive Fibonacci numbers. The Fibonacci matrix is $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

a. Find the eigenvalues of Q. They are $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$.
b. Find eigenvectors in terms of φ and ψ.
c. Use the diagonalisation to explain Binet's formula $F_n = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$.
d. Evaluate $\dfrac{\varphi^{10}}{\sqrt{5}}$ and compare with $F_{10} = 55$.

Question 14

Question 14 – A machine that breaks down *(Section B · Real-life practice)*

A machine is either working (W) or broken (B) at the start of each day. A working machine breaks with probability 0.05. A broken machine is repaired by the next day

with probability 0.4. The transition matrix is $T = \begin{pmatrix} 0.95 & 0.4 \\ 0.05 & 0.6 \end{pmatrix}$.

a. Find the eigenvalues of T.
b. Find the long-term fraction of days on which the machine works.
c. Find the machine state probabilities after 2 days if it works today.
d. How fast does the system approach its long-term behaviour?

Question 15

Question 15 – A recurrence from a matrix *(Section C · Problem solving)*

A culture of bacteria is counted each generation. The counts (in millions) satisfy $a_{n+2} = 5a_{n+1} - 6a_n$ with $a_0 = 0$ and $a_1 = 1$.

n	0	1	2	3	4
a_n	0	1	5	19	65

a. Check the entries in the table.
b. Write the recurrence as $\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$ and find A.
c. Find the eigenvalues of A and write $a_n = \alpha 2^n + \beta 3^n$. Find α, β.
d. Find a_{10} and the long-run ratio a_{n+1}/a_n.

Question 16

Question 16 – Principal axes of a stress pattern *(Section C · Problem solving)*

The stress energy in a metal plate is modelled by the quadratic form $Q(x, y) = 5x^2 + 4xy + 2y^2$, which can be written $Q = \mathbf{x}^T S \mathbf{x}$ with $S = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

Engineers look for the principal axes, along which the pattern is simplest.

a. Find the eigenvalues of S.
b. Find unit eigenvectors and check that they are perpendicular.
c. Write Q in the form $\lambda_1 u^2 + \lambda_2 v^2$ in the rotated coordinates (u, v).
d. Describe the curve $Q = 6$ and give the lengths of its semi-axes.

Question 17

Question 17 – Growth of matrix powers *(Section C · Problem solving)*

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. The table shows the top-left entry of A^n.

n	1	2	3	4
$(A^n)_{11}$	2	5	15	50

a. Check the entries for $n = 2$ and $n = 3$ by multiplying matrices.
b. Find the characteristic equation and the eigenvalues of A.
c. Use Cayley-Hamilton to show that $A^2 = 5A - 5I$ and then that $A^{-1} = \frac{1}{5}(5I - A)$.
d. Explain why the ratio of successive entries tends to the dominant eigenvalue and compute it.

Question 18

Question 18 – A parameter in a symmetric system *(Section C · Problem solving)*

Let $A_k = \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix}$, where k is a real constant.

a. Show that the eigenvalues are $k + 1$ and $k - 1$ for every k, and find the eigenvectors.
b. For which k is A_k invertible?
c. For which k is the system $\dot{\mathbf{x}} = A_k\mathbf{x}$ stable, so that all solutions tend to zero?
d. For $k = 0$ describe A_0 geometrically.

Question 19

Question 19 – Classifying prototypes *(Section C · Problem solving)*

Engineers describe four prototype control systems $\dot{\mathbf{x}} = A\mathbf{x}$ by the trace and determinant of A. For a 2×2 matrix, $\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$ with $\tau = \operatorname{tr} A$ and $\Delta = \det A$.

System	τ	Δ
P1	-5	6
P2	-2	5
P3	1	-2
P4	4	3

a. Find the eigenvalues of each system.
b. Classify each as stable node, stable spiral, saddle or unstable node.
c. Which systems tend to zero for every starting point?
d. For P2, write the form of the solutions and the oscillation frequency.

Question 20

Question 20 – Challenge: distinct eigenvalues give independent eigenvectors *(Section D · Challenge)*

Let $\mathbf{v}_1, \mathbf{v}_2$ be eigenvectors of a matrix A for distinct eigenvalues $\lambda_1 \neq \lambda_2$. Prove that they are linearly independent.

a. Suppose $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{0}$. Apply A to get a second equation.
b. Eliminate $\mathbf{v}_2$ and deduce $\alpha = 0$.
c. Deduce $\beta = 0$.
d. Show that A and A^T have the same eigenvalues, and that for a 2×2 matrix $\lambda_1 + \lambda_2 = \operatorname{tr} A$.