Answers - Eigenvalues and eigenvectors

Year: University Year 1

Answer to Question 1

a. Subtract λ from the diagonal and take the determinant: $\det(A-\lambda I) = (4-\lambda)(3-\lambda) - 2\times 1 = \lambda^2 - 7\lambda + 10$. This factorises as $(\lambda-2)(\lambda-5)=0$, so the eigenvalues are $\lambda = 2$ and $\lambda = 5$.

b. For $\lambda = 2$: $(A-2I)\mathbf{v} = \mathbf{0}$ gives $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}$. Both rows say $2x + y = 0$, so $y = -2x$ and we may take $\mathbf{v} = (1, -2)$. For $\lambda = 5$: $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\mathbf{v} = \mathbf{0}$ gives $x = y$, so $\mathbf{v} = (1, 1)$. Any non-zero multiple is also correct.

c. The trace is $4 + 3 = 7$, which equals $2 + 5$. The determinant is $12 - 2 = 10$, which equals 2×5. Both checks confirm the eigenvalues.

Answer to Question 2

a. The matrix is upper triangular, so the eigenvalues are the diagonal entries: **1 and 3**.

b. Expanding along the first row, $\det(A-\lambda I) = (2-\lambda)\left[(3-\lambda)(9-\lambda) - 16\right] = (2-\lambda)(\lambda^2 - 12\lambda + 11) = (2-\lambda)(\lambda-1)(\lambda-11)$. The eigenvalues are **1, 2 and 11**.

c. For a 2×2 matrix the characteristic equation is $\lambda^2 - (\mathrm{tr})\lambda + \det = 0$, here $\lambda^2 - 7\lambda + 10 = 0$, so the eigenvalues are **2 and 5**.

Answer to Question 3

a. $\det(A-\lambda I) = \lambda^2 + 1 = 0$, so $\lambda = \pm i$. For $\lambda = i$: $-ix - y = 0$, so $y = -ix$ and $\mathbf{v} = (1, -i)$. A real rotation has no real direction left unchanged, hence the complex eigenvalues.

b. $(1-\lambda)^2 + 4 = 0$ gives $1 - \lambda = \pm 2i$, so $\lambda = 1 \mp 2i$, that is $1 + 2i$ **and** $1 - 2i$. For $\lambda = 1 + 2i$: $-2ix - 2y = 0$ so $y = -ix$ and $\mathbf{v} = (1, -i)$. The eigenvalues come in a conjugate pair, with modulus $\sqrt{1+4} = \sqrt{5}$: the matrix is a rotation combined with an enlargement by $\sqrt{5}$.

Answer to Question 4

a. The characteristic equation is $(2-\lambda)^2 - 1 = 0$, so $2 - \lambda = \pm 1$ and $\lambda = 1$ or 3. For $\lambda = 3$: $-x + y = 0$, so $x = y$ and $\mathbf{v} = (1, 1)$. For $\lambda = 1$: $x + y = 0$, so $\mathbf{v} = (1, -1)$.

b. Place the eigenvectors as columns of P in the same order as the eigenvalues in D: $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\det P = -2$, $P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

c. Powers of a diagonal matrix are easy: $A^5 = PD^5P^{-1}$ with $D^5 = \begin{pmatrix} 243 & 0 \\ 0 & 1 \end{pmatrix}$. Multiplying out, $A^5 = \frac{1}{2}\begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix} = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$. Check: the trace $244 = 3^5 + 1^5$.

Answer to Question 5

a. $(2-\lambda)^2 = 0$, so $\lambda = 2$ with **algebraic multiplicity 2**.

b. $(N - 2I)\mathbf{v} = \mathbf{0}$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0}$ gives $y = 0$, so all eigenvectors are multiples of $(1, 0)$. Only one independent eigenvector: **geometric multiplicity 1**.

c. Diagonalisation needs two independent eigenvectors in $\mathbb{R}^2$. N has only one, so it is not diagonalisable.

Answer to Question 6

a. Let $A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$. Applying A again, $A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$. Multiplying $A\mathbf{v} = \lambda\mathbf{v}$ by A^{-1} and dividing by $\lambda \neq 0$ gives $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Finally $(A + 3I)\mathbf{v} = (\lambda + 3)\mathbf{v}$. So A^2 has eigenvalues **4 and 25**; A^{-1} has $\frac{1}{2}$ **and** $\frac{1}{5}$; $A + 3I$ has **5 and 8**.

b. The determinant is the product of the eigenvalues and the trace is their sum, so $\det(A^2) = 4 \times 25 = 100$ and $\operatorname{tr}(A^2) = 4 + 25 = 29$. Directly, $A^2 = \begin{pmatrix} 18 & 7 \\ 14 & 11 \end{pmatrix}$, with trace 29 and determinant $198 - 98 = 100$. The two methods agree.

Answer to Question 7

a. Trace $= 1.7$ and determinant $= 0.72 - 0.02 = 0.70$. The characteristic equation $\lambda^2 - 1.7\lambda + 0.7 = 0$ gives $(\lambda - 1)(\lambda - 0.7) = 0$: eigenvalues **1 and 0.7**.

b. For $\lambda = 1$: $-0.1a + 0.2b = 0$ gives $a = 2b$, $\mathbf{v} = (2, 1)$. For $\lambda = 0.7$: $0.2a + 0.2b = 0$ gives $\mathbf{v} = (1, -1)$.

c. The eigenvalue 1 eigenvector scaled to total 1 is $\left(\frac{2}{3}, \frac{1}{3}\right)$: in the long run **A has 66.7%** and B has 33.3%.

d. Write $(1, 0) = \frac{1}{3}(2, 1) + \frac{1}{3}(1, -1)$. Then $\mathbf{v}_n = \frac{1}{3}(2, 1) + \frac{1}{3}(0.7)^n(1, -1)$. For $n = 5$: $0.7^5 = 0.16807$, so $a_5 = \frac{2}{3} + \frac{1}{3}(0.16807) = 0.7227$: about **72.3%**.

Answer to Question 8

a. The characteristic equation is $\det(L - \lambda I) = -\lambda(0.8 - \lambda) - 2 \times 0.5 = \lambda^2 - 0.8\lambda - 1 = 0$. By the quadratic formula, $\lambda = \dfrac{0.8 \pm \sqrt{0.64 + 4}}{2} = 0.4 \pm 1.077$, so the eigenvalues are **1.477 and** -0.677.

b. The eigenvalue with the largest modulus dominates, so long-term behaviour is governed by 1.477. Each period the population is multiplied by about 1.477, a growth of about **47.7% per period**.

c. For $\lambda = 1.477$, the first row of $(L - \lambda I)\mathbf{v} = \mathbf{0}$ gives $-1.477J + 2A = 0$. So in the long run $A/J = 1.477/2 \approx$ **0.739**: about 74 adults for every 100 juveniles.

d. Writing the starting vector in terms of the two eigenvectors, the second part is multiplied by $(-0.677)^n$. Since $|-0.677| < 1$, this tends to zero, so only the dominant behaviour remains.

Answer to Question 9

a. $(2 - \mu)^2 - 1 = 0$ gives $\mu = 1$ or $\mu = 3$.

b. For $\mu = 1$: $\mathbf{v} = (1, 1)$ — the carriages move together, so the middle spring is not stretched. For $\mu = 3$: $\mathbf{v} = (1, -1)$ — they move in opposite directions.

c. $\dfrac{k}{m} = 100$. So $\omega^2 = 100\mu$: $\omega = 10$ rad/s for $\mu = 1$ and $\omega = \sqrt{300} \approx 17.3$ rad/s for $\mu = 3$.

d. The opposite-direction mode is faster because the middle spring is stretched and compressed as well, so the restoring force is larger.

Answer to Question 10

a. The characteristic equation is $(3 - \lambda)^2 - 1 = 0$, so $3 - \lambda = \pm 1$, which gives $\lambda = 2$ or 4.

b. For $\lambda = 4$: $-x + y = 0$, so $\mathbf{v} = (1, 1)$. For $\lambda = 2$: $x + y = 0$, so $\mathbf{v} = (1, -1)$. Vectors along these lines are mapped to multiples of themselves; they are the two diagonals of the picture.

c. Points on the diagonal $y = x$ move **4** times further from the centre, and points on $y = -x$ move **2** times further. Every other direction is both stretched and turned.

d. The area scale factor is $|\det S| = 9 - 1 = 8$, and $4 \times 2 = 8$. Any shape in the photo has its area multiplied by 8.

Answer to Question 11

a. $(-3 - \lambda)^2 - 1 = 0$ gives $\lambda = -2$ and $\lambda = -4$. Eigenvectors: $(1, 1)$ for -2, and

$(1, -1)$ for -4.

b. $\mathbf{x}(t) = c_1 e^{-2t}(1, 1) + c_2 e^{-4t}(1, -1)$.

c. At $t = 0$: $c_1 + c_2 = 4$ and $c_1 - c_2 = 0$, so $c_1 = c_2 = 2$. After 1 hour: $x_1 = 2e^{-2} + 2e^{-4} \approx 0.307$ and $x_2 = 2e^{-2} - 2e^{-4} \approx 0.234$.

d. Both eigenvalues are negative, so e^{-2t} and e^{-4t} tend to 0: the deviations vanish and the system is stable.

Answer to Question 12

a. Column A: $0 + \frac{1}{2} + \frac{1}{2}$; column B: $0 + 0 + 1$; column C: $1 + 0 + 0$. All equal 1.

b. The equations are $r_A = r_C$, $r_B = \frac{1}{2} r_A$, $r_C = \frac{1}{2} r_A + r_B$. So $r_B = \frac{1}{2} r_A$ and $r_C = r_A$. With the sum equal to 1: $r_A \left(1 + \frac{1}{2} + 1\right) = 1$, so $r_A = 0.4$, $r_B = 0.2$, $r_C = 0.4$.

c. Pages A and C tie with **0.4** each; B has 0.2.

d. Page C receives all of B's importance as well as half of A's; B receives only half of A's.

Answer to Question 13

a. The characteristic equation is $(1 - \lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$. By the quadratic formula, $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$, so the eigenvalues are φ and ψ.

b. The first row of $(Q - \lambda I)\mathbf{v} = \mathbf{0}$ is $(1 - \lambda)x + y = 0$. Using $\lambda^2 = \lambda + 1$, the vector $\mathbf{v} = (\lambda, 1)$ satisfies both rows, so the eigenvectors are $(\varphi, 1)$ and $(\psi, 1)$.

c. Since $Q = PDP^{-1}$ with $D = \operatorname{diag}(\varphi, \psi)$, we get $Q^n = PD^nP^{-1}$. Multiplying out, the off-diagonal entry is $\dfrac{\varphi^n - \psi^n}{\varphi - \psi}$, and $\varphi - \psi = \sqrt{5}$. This entry is F_n, which proves Binet's formula.

d. $\varphi^{10} \approx 122.99$, and $122.99 / 2.236 \approx 55.0$, matching $F_{10} = 55$. Because $|\psi| < 1$, $\psi^n \to 0$, so F_n is the nearest integer to $\varphi^n / \sqrt{5}$.

Answer to Question 14

a. Trace 1.55, determinant $0.57 - 0.02 = 0.55$: $\lambda^2 - 1.55\lambda + 0.55 = 0$ gives $(\lambda - 1)(\lambda - 0.55) = 0$.

b. For $\lambda = 1$: $-0.05w + 0.4b = 0$, so $w = 8b$. With $w + b = 1$: $w = \frac{8}{9} \approx$ **0.889**.

c. $\mathbf{v}_1 = (0.95, 0.05)$, $\mathbf{v}_2 = T\mathbf{v}_1 = (0.9025 + 0.02,\ 0.0475 + 0.03) = (0.9225, 0.0775)$.

d. The second eigenvalue is 0.55, so the distance from the steady state shrinks by a factor of 0.55 each day: after 5 days it is $0.55^5 \approx 5\%$ of its original size.

Answer to Question 15

a. Using the recurrence: $a_2 = 5(1) - 6(0) = 5$, $a_3 = 5(5) - 6(1) = 19$ and $a_4 = 5(19) - 6(5) = 65$. The table is correct.

b. The top row must give $a_{n+1} = 5a_n - 6a_{n-1}$ and the bottom row simply copies a_n, so $A = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}$.

c. The characteristic equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$, giving eigenvalues 2 and 3, so $a_n = \alpha 2^n + \beta 3^n$. From $a_0 = \alpha + \beta = 0$ and $a_1 = 2\alpha + 3\beta = 1$ we find $\beta = 1$ and $\alpha = -1$. Hence $a_n = 3^n - 2^n$. Check: $a_4 = 81 - 16 = 65$.

d. $a_{10} = 59049 - 1024 = 58\,025$ million. Since $a_{n+1}/a_n = \dfrac{3^{n+1} - 2^{n+1}}{3^n - 2^n}$ and $(2/3)^n \to 0$, the ratio tends to 3, the dominant eigenvalue.

Answer to Question 16

a. $\lambda^2 - 7\lambda + 6 = 0$ gives $\lambda = 1$ and $\lambda = 6$.

b. For $\lambda = 6$: $-x + 2y = 0$, so $(2, 1)/\sqrt{5}$. For $\lambda = 1$: $4x + 2y = 0$, so $(1, -2)/\sqrt{5}$. The dot product is $2 - 2 = 0$, so they are perpendicular (a general property of symmetric matrices).

c. $Q = 6u^2 + v^2$, where u is measured along $(2, 1)/\sqrt{5}$ and v along $(1, -2)/\sqrt{5}$.

d. $Q = 6$ becomes $u^2 + \dfrac{v^2}{6} = 1$ — an ellipse. The semi-axis along the u-direction is 1 and along the v-direction is $\sqrt{6} \approx 2.45$.

Answer to Question 17

a. Multiplying, $A^2 = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}$; then $A^3 = A^2 A = \begin{pmatrix} 15 & 20 \\ 20 & 35 \end{pmatrix}$. The top-left entries are 5 and 15, as in the table.

b. With trace 5 and determinant $6 - 1 = 5$, the characteristic equation is $\lambda^2 - 5\lambda + 5 = 0$, so $\lambda = \dfrac{5 \pm \sqrt{5}}{2}$, about 3.618 and 1.382.

c. By Cayley-Hamilton, A satisfies its own characteristic equation: $A^2 - 5A + 5I = 0$, so $A^2 = 5A - 5I$. Rearranging, $A(5I - A) = 5I$, hence $A^{-1} = \dfrac{1}{5}(5I - A) = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$.

d. Diagonalising, $A^n = \lambda_1^n P_1 + \lambda_2^n P_2$ for fixed matrices P_1, P_2. As n grows, λ_1^n dominates, so $\dfrac{(A^{n+1})_{11}}{(A^n)_{11}} \to \lambda_1 \approx 3.618$. The table gives $50/15 \approx 3.33$, already

close.

Answer to Question 18

a. $(k-\lambda)^2 - 1 = 0$ gives $\lambda = k \pm 1$. The eigenvector for $k+1$ is $(1,1)$ and for $k-1$ is $(1,-1)$, independent of k.
b. $\det = k^2 - 1 \neq 0$, so A_k is invertible exactly when $k \neq \pm 1$ (the eigenvalues are non-zero).
c. Solutions behave like $e^{(k+1)t}$ and $e^{(k-1)t}$, which both tend to zero only if $k+1$, so k .
d. $A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the reflection in $y = x$. Its eigenvalues are ± 1: it fixes $(1,1)$ and reverses $(1,-1)$.

Answer to Question 19

a. P1: $\sqrt{25-24} = 1$, so $\lambda = -2, -3$. P2: $\sqrt{4-20} = 4i$, so $\lambda = -1 \pm 2i$. P3: $\sqrt{1+8} = 3$, so $\lambda = 2, -1$. P4: $\sqrt{16-12} = 2$, so $\lambda = 3, 1$.
b. P1 **stable node**; P2 **stable spiral**; P3 **saddle** (Δ); P4 **unstable node**.
c. P1 and P2: all eigenvalues have negative real part.
d. P2: $\mathbf{x} = e^{-t}(\mathbf{a}\cos 2t + \mathbf{b}\sin 2t)$, a decaying oscillation with angular frequency 2 (period $\pi \approx 3.14$).

Answer to Question 20

a. Applying A: $\alpha\lambda_1\mathbf{v}_1 + \beta\lambda_2\mathbf{v}_2 = \mathbf{0}$.
b. Multiply the original equation by λ_2 and subtract: $\alpha(\lambda_1 - \lambda_2)\mathbf{v}_1 = \mathbf{0}$. Since $\mathbf{v}_1 \neq \mathbf{0}$ and $\lambda_1 \neq \lambda_2$, $\alpha = 0$.
c. Then $\beta\mathbf{v}_2 = \mathbf{0}$ with $\mathbf{v}_2 \neq \mathbf{0}$ gives $\beta = 0$. So the vectors are independent.
d. $\det(A^T - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A - \lambda I)$, so the characteristic polynomials agree. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the characteristic polynomial is $\lambda^2 - (a+d)\lambda + (ad - bc)$, so the sum of its roots is $a + d = \operatorname{tr} A$.