Test answers - Differentiation and its applications

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

a. Quotient rule: $f'(x) = \dfrac{1 \cdot (e^x + 1) - x \cdot e^x}{(e^x + 1)^2}$ (1) $= \dfrac{e^x + 1 - xe^x}{(e^x + 1)^2}$ (1).

b. $\ln y = x \ln \sin x$ (1). So $\dfrac{y'}{y} = \ln \sin x + x \cot x$, and $y' = (\sin x)^x \left(\ln \sin x + x \cot x\right)$ (1).

In (a), the quotient rule is applied with $u = x$ and $v = e^x + 1$: $f' = \frac{u'v - uv'}{v^2}$, and a common error is to reverse the order of the terms in the numerator, which changes the sign. In (b), the power rule $x \cdot (\sin x)^{x-1}$ is wrong because the exponent is not constant; logarithmic differentiation is needed, and the condition $0 < x < \pi$ guarantees $\sin x > 0$ so that $\ln \sin x$ is defined.

Answer to Question 2 (4 marks)

a. The side along the river is $120 - 2x$, so $A = x(120 - 2x)$ (1).
b. $A' = 120 - 4x = 0$ gives $x = 30$ (1). $A'' = -4$, so a maximum (1).
c. $A = 30 \times 60 = $ **1 800 m**^{2} (1).

In (a), the fence is used for two sides of length x and one side parallel to the river, so the third side is $120 - 2x$; the river needs no fence. In (b), the second derivative test confirms the nature of the stationary point; the endpoints $x = 0$ and $x = 60$ give zero area, which also confirms a maximum. The pen measures 30 m by 60 m.

Answer to Question 3 (4 marks)

a. f is a polynomial, so it is continuous on $[0, 2]$ and differentiable on $(0, 2)$ (1).

b. The average gradient is $\dfrac{8 - 0}{2} = 4$ (1). $f'(c) = 3c^2 = 4$ (1), so $c = \dfrac{2}{\sqrt{3}} \approx 1.155$, which lies in $(0, 2)$ (1).

In (a), both hypotheses must be stated: continuity on the closed interval and differentiability on the open interval. In (b), the theorem guarantees a point where the tangent is parallel to the chord joining $(0, 0)$ and $(2, 8)$. The negative root $-\frac{2}{\sqrt{3}}$ is rejected because it lies outside $(0, 2)$.

Answer to Question 4 (4 marks)

a. Form $\dfrac{0}{0}$ (1). Apply the rule three times: $\dfrac{\cos x - 1}{3x^2} \to \dfrac{-\sin x}{6x} \to \dfrac{-\cos x}{6}$ (1), giving $-\dfrac{1}{6}$ (1).

b. $1 - \cos x = \dfrac{x^2}{2} - \dfrac{x^4}{24} + \cdots$, so the ratio tends to $\dfrac{1}{2}$ (1).

In (a), l'Hôpital's rule may be used each time only because the form is still $\frac{0}{0}$; check this at every step. Series give the same answer: $\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots$, so dividing by x^3 gives $-\frac{1}{6}$ in the limit. In (b), dividing the series by x^2 leaves $\frac{1}{2} - \frac{x^2}{24} + \cdots$.

Answer to Question 5 (4 marks)

$f(4) = 2$, $f'(x) = \frac{1}{2}x^{-1/2}$ gives $\frac{1}{4}$, $f''(x) = -\frac{1}{4}x^{-3/2}$ gives $-\frac{1}{32}$ (2). So $T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ (1). (b) $T_2(4.4) = 2 + 0.1 - 0.0025 = 2.0975$ (1), within 0.0002 of 2.09762.

The coefficient of $(x-4)^2$ is $\frac{f''(4)}{2!} = -\frac{1}{64}$; a common error is to forget the factorial and write $-\frac{1}{32}$. In (b), the error is about 0.00012, which is small because 4.4 is close to the centre 4. The next term of the series is positive, which explains why the estimate is slightly too small.