Lesson - Differentiation and its applications

Year: University Year 1

Differentiation measures instantaneous rate of change. At university it is defined as a limit, and its key theorems (Rolle, the mean value theorem, Taylor) support optimisation, limits and approximation.

Starter activity

Drive 60 miles in exactly one hour. Must you have been going at exactly 60 mph at some instant, even if you sometimes went faster and sometimes slower? Sketch a graph of distance against time that shows your answer. Then ask: how would you change your sketch if the journey were 60 miles in 45 minutes, and the speed limit were 70 mph?

The derivative

Definition: derivative

The **derivative** of f at a is $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, provided the limit exists.

Geometrically it is the gradient of the tangent at $(a, f(a))$; analytically it is the best linear approximation to f near a.

Property: differentiable implies continuous

If $f'(a)$ exists then f is continuous at a. The converse is false: $|x|$ is continuous at 0 but not differentiable there, because the one-sided limits of the difference quotient are -1 and 1.

Example 1: first principles

Differentiate $f(x) = x^3$ from first principles.

$\dfrac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$ as $h \to 0$.

Rules of differentiation

Rule: product, quotient and chain rules

$(uv)' = u'v + uv'$, $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $(f \circ g)'(x) = f'(g(x))\,g'(x)$. Standard derivatives: $(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = 1/x$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\arctan x)' = \frac{1}{1+x^2}$, $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

Method: logarithmic and implicit differentiation

1. For $y = f(x)^{g(x)}$, take logarithms: $\ln y = g(x)\ln f(x)$.
2. Differentiate both sides with respect to x, using the chain rule on $\ln y$.
3. For an implicit curve, differentiate every term, with $\frac{d}{dx}y^2 = 2yy'$, then solve for y'.
4. The inverse function rule is $(f^{-1})'(y) = 1/f'(x)$ where $y = f(x)$.

Example 2

Differentiate $y = x^x$, and find y' on $x^2 + y^2 = 25$.

$\ln y = x\ln x \Rightarrow y'/y = \ln x + 1$, so $y' = x^x(\ln x + 1)$. For the circle, $2x + 2y\,y' = 0$, so $y' = -x/y$.

Stationary points and optimisation

Rule: second derivative test

A point with $f'(c) = 0$ is **stationary**. If $f''(c) > 0$ it is a local minimum; if $f''(c)$ a local maximum. If $f''(c) = 0$ the test is inconclusive: examine the sign of f' either side.

Example 3

Find and classify the stationary points of $f(x) = x^3 - 3x$.

$f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. Since $f''(x) = 6x$, $f''(1) = 6 > 0$: local minimum, $f(1) = -2$; and $f''(-1) = -6$: local maximum, $f(-1) = 2$.

Example 4: optimisation

Among rectangles of perimeter 20, find the greatest area.

Sides x and $10 - x$: $A = x(10 - x)$, $A' = 10 - 2x = 0$ at $x = 5$, and $A'' = -2$. The maximum area is 25, a square.

Rolle's theorem and the mean value theorem

Theorem: Rolle's theorem

If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$, then $f'(c)=0$ for some $c\in(a,b)$.

Theorem: mean value theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), there is $c\in(a,b)$ with $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Note: consequences

If $f'>0$ on an interval then f is strictly increasing; if $f'=0$ throughout, f is constant. The MVT also gives inequalities such as $|\sin a-\sin b|\le|a-b|$, because $|\cos c|\le 1$.

L'Hopital's rule

Rule: L'Hopital's rule

If $f(a)=g(a)=0$ (or both tend to $\pm\infty$) and $\lim f'/g'$ exists, then $\lim f/g=\lim f'/g'$.

Example 5

Evaluate $\lim_{x\to0}\dfrac{e^x-1-x}{x^2}$ and $\lim_{x\to0^+}x\ln x$.

First: $\lim\frac{e^x-1}{2x}=\lim\frac{e^x}{2}=\frac12$. Second: write $x\ln x=\frac{\ln x}{1/x}$, which gives $\frac{1/x}{-1/x^2}=-x\to0$.

Taylor polynomials

Definition: Taylor polynomial

If f has n derivatives at a, its **Taylor polynomial** of degree n is $T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$.

Theorem: Taylor's theorem

$f(x)=T_n(x)+R_n(x)$ with Lagrange remainder $R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$ for some ξ between a and x. Maclaurin series ($a=0$): $e^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$, $\sin x=x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots$, $\cos x=1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots$, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$.

Example 6

Estimate $e^{0.1}$.

$e^{0.1} \approx 1 + 0.1 + 0.005 + 0.000167 = 1.105167$; the true value is 1.105171.

Watch out

- Forgetting the inner derivative in the chain rule.
- Quoting the MVT without checking differentiability.
- Applying L'Hopital when the limit is not indeterminate.
- Forgetting the factorial in Taylor coefficients.

Reasoning with derivatives

Method: optimisation on a closed interval

1. Write the quantity to be maximised or minimised as a function of one variable, using any constraint to eliminate the others.
2. State the domain, including its end points.
3. Solve $f'(x) = 0$ and keep only the solutions inside the domain.
4. Compare f at every stationary point and at both end points. Use the second derivative or a sign table to classify the stationary points.
5. Answer the original question in context, with units.

Worked example 7: The quotient rule with simplification

Differentiate $y = \dfrac{\ln x}{x^2}$ and find where $y' = 0$.

Take $u = \ln x$ and $v = x^2$. Then $y' = \dfrac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4} = \dfrac{x - 2x \ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$.

Setting the numerator to zero gives $\ln x = \frac{1}{2}$, so $x = \sqrt{e} \approx 1.65$. Because y' is positive before this point and negative after it, the function has a maximum there, with value $\dfrac{1/2}{e} = \dfrac{1}{2e} \approx 0.184$.

Worked example 8: A tangent to an implicit curve

Find the tangent to $x^2 + xy + y^2 = 7$ at $(2, 1)$.

Differentiating implicitly gives $2x + y + x\,y' + 2y\,y' = 0$. At $(2,1)$: $4 + 1 + 2y' + 2y' = 0$, so $y' = -\frac{5}{4}$. The tangent is $y - 1 = -\frac{5}{4}(x - 2)$, that is $5x + 4y = 14$. As a check, the point lies on the curve because $4 + 2 + 1 = 7$. Notice that we never needed to solve for y in terms of x.

Worked example 9: Using the mean value theorem to prove an inequality

Prove that $\ln(1+x)$ for $x>0$.

Apply the mean value theorem to $f(t)=\ln(1+t)$ on $[0,x]$. It is continuous and differentiable, so there is a number c between 0 and x with $\dfrac{\ln(1+x)-\ln 1}{x}=\dfrac{1}{1+c}$. Since $c>0$ we have $\dfrac{1}{1+c}$, and so $\ln(1+x)$. The same method proves many inequalities: choose a function, apply the theorem on a suitable interval, and bound the derivative at the unknown point.

Worked example 10: Composite Maclaurin series

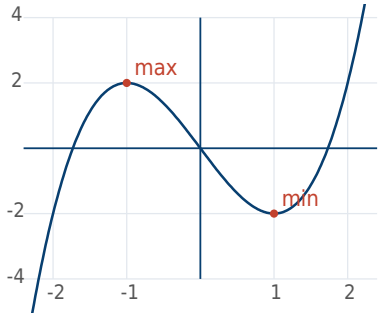
Find the Maclaurin series of $\dfrac{\sin x}{x}$ up to x^4 and use it to evaluate $\lim_{x\to 0}\dfrac{\sin x}{x}$.

Divide the series for $\sin x$ by x: $\dfrac{\sin x}{x}=1-\dfrac{x^2}{6}+\dfrac{x^4}{120}-\cdots$. As $x\to 0$ every term except the first vanishes, so the limit is 1. This is more informative than l'Hôpital's rule because it also tells us that near zero $\dfrac{\sin x}{x}\approx 1-\dfrac{x^2}{6}$, so the function dips below 1 as x moves away from 0.

Worked example 11: A limit of the form infinity minus infinity

Evaluate $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Combine the fractions: $\dfrac{\sin x-x}{x\sin x}$, which has the form $\dfrac{0}{0}$. Differentiating top and bottom gives $\dfrac{\cos x-1}{\sin x+x\cos x}$, still $\dfrac{0}{0}$. Once more: $\dfrac{-\sin x}{2\cos x-x\sin x}\to\dfrac{0}{2}=0$. So the limit is 0. Rewriting the expression so that it is a single quotient is the key step whenever you meet $\infty-\infty$.



Method: related rates

1. Draw a diagram and name every changing quantity with a letter, writing down

which rates are known and which one is wanted.
2. Find an equation linking the quantities that is true at every moment, such as Pythagoras or a volume formula.
3. Differentiate both sides with respect to time t, using the chain rule on every variable, for example $\frac{d}{dt}r^2 = 2r\frac{dr}{dt}$.
4. Only now substitute the particular values, and solve for the unknown rate.
5. Interpret the sign: a negative rate means the quantity is decreasing.

Worked example 12: The inverse function rule

Show that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.

Let $y = \arctan x$, so $x = \tan y$. Then $\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$. The inverse function rule says that $\frac{dy}{dx} = \frac{1}{dx/dy}$, so $\frac{dy}{dx} = \frac{1}{1+x^2}$. The same idea gives the derivatives of $\arcsin x$ and $\ln x$ from those of $\sin y$ and e^y.

Worked example 13: Concavity and the second derivative

Find the intervals on which $f(x) = x^4 - 6x^2$ is concave up, and find its points of inflection.

The first derivative is $f'(x) = 4x^3 - 12x$ and the second is $f''(x) = 12x^2 - 12 = 12(x-1)(x+1)$. The second derivative is positive when $|x| > 1$, so f is concave up on $(-\infty, -1)$ and $(1, \infty)$, and negative on $(-1, 1)$, where f is concave down. The sign of f'' changes at $x = \pm 1$, so these are points of inflection, at $(\pm 1, -5)$. A change of curvature is the signature of an inflection point.

Worked example 14: Estimating with a linear approximation

Use a tangent line to estimate $\sqrt{101}$ and state whether the estimate is too large or too small.

Take $f(x) = \sqrt{x}$ near $a = 100$. Then $f(100) = 10$ and $f'(100) = \frac{1}{2\sqrt{100}} = 0.05$, so $\sqrt{101} \approx 10 + 0.05 \times 1 = 10.05$. Since $f''(x) = -\frac{1}{4}x^{-3/2}$ the graph is concave down, so the tangent lies above the curve and the estimate is slightly too large. The true value is 10.04988, and the error of about 0.0001 is just what the second-order term $-\frac{1}{8000}$ predicts.

Worked example 15: Differentiability at a point

Is $f(x) = x|x|$ differentiable at $x = 0$? Use the definition.

Compute the difference quotient at 0: $\dfrac{f(h)-f(0)}{h} = \dfrac{h|h|}{h} = |h|$. As $h \to 0$ from either side, $|h| \to 0$, so the limit exists and $f'(0) = 0$. The function is therefore differentiable at 0, even though it is defined using the absolute value. Compare this with $|x|$ itself, whose difference quotient is $\dfrac{|h|}{h}$, equal to 1 for positive h and -1 for negative h, so no limit exists. Checking the one-sided limits is the safest way to decide.

Note: when the second derivative test fails

If $f'(c) = 0$ and $f''(c) = 0$ the test gives no information. The point can be a minimum (as for x^4 at 0), a maximum (as for $-x^4$), or a point of inflection (as for x^3). In such cases examine the sign of f' on either side of c, or look at the first non-zero higher derivative.

Watch out: hypotheses matter

- The mean value theorem fails for $f(x) = |x|$ on $[-1, 2]$, because f is not differentiable at 0. Always check continuity on the closed interval and differentiability on the open one.
- L'Hôpital's rule applies only to the forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$. For example $\lim_{x \to 0} \dfrac{x+1}{x+2} = \dfrac{1}{2}$, whereas applying the rule would wrongly give 1.
- A zero derivative does not guarantee a turning point, and a maximum on an interval may occur at an end point.

Did you know?

The derivative was developed independently by Isaac Newton and Gottfried Leibniz in the seventeenth century, and the quarrel over who was first lasted for decades. The rule we call l'Hôpital's rule was actually discovered by the Swiss mathematician Johann Bernoulli, who sold it to the Marquis de l'Hôpital in 1694 in exchange for a salary, and the Marquis published it in the first calculus textbook.

Otto's tip

Always state the hypotheses of a theorem before you use it.