Worksheet - Differentiation and its applications

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Differentiation rules *(Section A · Skills practice)*

Differentiate each function, simplifying where possible.

a. $f(x) = x^3 \ln x$ [2 marks]
b. $g(x) = e^{2x} \sin x$ [2 marks]
c. $h(x) = \dfrac{x^2 + 1}{x - 1}$ [3 marks]
d. $k(x) = \sqrt{1 + x^2}$ [2 marks]

Question 2

Question 2 – Implicit and logarithmic differentiation *(Section A · Skills practice)*

Find the derivative asked for in each part.

a. $\dfrac{dy}{dx}$ for $y = x^x$, $x > 0$ [3 marks]
b. $\dfrac{dy}{dx}$ at $(3, 4)$ on the circle $x^2 + y^2 = 25$ [3 marks]
c. The equation of the tangent to $x^3 + y^3 = 9xy$ at $(2, 4)$ is wanted: find $\dfrac{dy}{dx}$ there. [4 marks]

Question 3

Question 3 – First principles *(Section A · Skills practice)*

Use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate each function.

a. $f(x) = x^3$ [3 marks]
b. $f(x) = \dfrac{1}{x}$ for $x \neq 0$ [3 marks]
c. $f(x) = \sqrt{x}$ for $x > 0$ [3 marks]

Question 4

Question 4 – Stationary points *(Section A · Skills practice)*

Find and classify the stationary points of each function using the second derivative.

a. $f(x) = x^3 - 3x$ [4 marks]
b. $g(x) = x^4 - 4x^3$ [5 marks]

Question 5

Question 5 – Limits with l'Hôpital's rule *(Section A · Skills practice)*

Evaluate each limit. State the form of each limit before you apply l'Hôpital's rule.

a. $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$ [3 marks]

b. $\lim_{x \to 0^+} x \ln x$ [3 marks]

c. $\lim_{x \to \infty} \dfrac{x^2}{e^x}$ [3 marks]

Question 6

Question 6 – Maclaurin polynomials *(Section A · Skills practice)*

Find the Maclaurin polynomial of the stated degree for each function.

a. $\cos x$ up to x^4 [3 marks]
b. $\ln(1 + x)$ up to x^3 [3 marks]
c. e^{-x^2} up to x^4, by substituting into the series for e^u [3 marks]

Question 7

Question 7 – The largest open box *(Section B · Real-life practice)*

A sheet of card measures 30 cm by 20 cm. A square of side x cm is cut from each corner and the sides are folded up to make an open box.

a. Show that the volume is $V = x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$.
b. State the range of values of x that make sense.
c. Find the value of x that maximises V, to 2 decimal places, and justify that it is a maximum.
d. Find the maximum volume.

Question 8

Question 8 – Designing a drinks can *(Section B · Real-life practice)*

A drinks company wants a closed cylindrical can of volume 330 cm^3 using as little aluminium as possible. Let the radius be r cm and the height h cm.

a. Write h in terms of r and show that the surface area is $S = 2\pi r^2 + \dfrac{660}{r}$.
b. Find the value of r that minimises S.
c. Show that the best height equals the diameter, and find h.
d. Find the minimum surface area. Typical cans are taller and thinner than this; suggest a reason.

Question 9

Question 9 – Maximising profit *(Section B · Real-life practice)*

A company makes x phone cases per week. The price it can charge is $p = 100 - 0.02x$ pounds per case (see the table), and the weekly cost in pounds is $C = 2000 + 20x + 0.005x^2$.

Cases per week x	500	1 000	1 600	2 000
Price per case £	90	80	68	60

a. Write the weekly revenue $R = px$ in pounds and show that the profit is $P = 80x - 0.025x^2 - 2000$.
b. Find the number of cases that gives the greatest profit.
c. Find the greatest profit and the price charged then.
d. Explain why making more than the optimal number reduces profit.

Question 10

Question 10 – Speed cameras and the mean value theorem *(Section B · Real-life practice)*

Two speed cameras on a motorway are exactly 10 miles apart. A car passes the first at 10:00 and the second at 10:08. The position of the car (in miles from the first camera) at time t minutes is a differentiable function $s(t)$.

a. Find the car's average speed in miles per hour.
b. The speed limit is 70 mph. Use the mean value theorem to show that the car must have been travelling faster than 70 mph at some moment.
c. State exactly which hypotheses of the theorem you used.
d. Would the same argument work if the camera operator only knew the car passed each camera, but not that it moved continuously? Explain.

Question 11

Question 11 – The sliding ladder *(Section B · Real-life practice)*

A ladder 5 m long leans against a vertical wall. The foot of the ladder is pulled away

from the wall at 0.5 m/s. Let x be the distance of the foot from the wall and y the height of the top.

a. Explain why $x^2 + y^2 = 25$ and deduce that $x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$.
b. Find $\dfrac{dy}{dt}$ when $x = 3$.
c. Find the speed of the top when $x = 4$.
d. What happens to the speed of the top as $x \to 5$? Is the model realistic?

Question 12

Question 12 – Inflating a balloon *(Section B · Real-life practice)*

A spherical balloon is inflated at a constant rate of 100 cm^3 per second. Recall $V = \dfrac{4}{3}\pi r^3$ and surface area $S = 4\pi r^2$.

a. Show that $\dfrac{dr}{dt} = \dfrac{25}{\pi r^2}$.
b. Find the rate of increase of the radius when $r = 10$ cm.
c. Show that $\dfrac{dS}{dt} = \dfrac{200}{r}$ and find the rate at which the surface area grows when $r = 10$.
d. What does the formula in (c) say about the balloon as it gets bigger?

Question 13

Question 13 – Fuel economy *(Section B · Real-life practice)*

A car's fuel consumption in litres per 100 km at speed v km/h is modelled by $F(v) = 0.002v^2 - 0.3v + 15$ for $40 \le v \le 120$. Petrol costs £1.50 per litre.

a. Find the speed that minimises fuel consumption and the minimum value of F.
b. Find the fuel used and the cost of a 300 km journey at that speed.
c. Find the cost of the same journey at 120 km/h.
d. How much money is saved by driving at the economical speed? How long does each journey take?

Question 14

Question 14 – A model rocket *(Section B · Real-life practice)*

A model rocket is launched upwards from the ground. Ignoring air resistance, its height after t seconds is $h(t) = 40t - 4.9t^2$ metres.

a. Find the velocity $v = h'(t)$ and the acceleration $a = h''(t)$.
b. Find when the rocket is at its highest point and the maximum height, to 1 decimal place.

c. Find when the rocket lands, and its speed on landing.
d. Show that the mean value theorem predicts a moment with zero velocity during the flight, and check it agrees with part (b).

Question 15

Question 15 – How good is the Taylor polynomial? *(Section C · Problem solving)*

A calculator estimates $e^{0.5}$ using the Maclaurin polynomial $T_n(x) = \sum_{k=0}^{n} \dfrac{x^k}{k!}$.

Lagrange's remainder says $|R_n(x)| \le \dfrac{e^{\xi}|x|^{n+1}}{(n+1)!}$ for some ξ between 0 and x.

n	$T_n(0.5)$
1	1.5
2	1.625
3	1.64583
4	1.64844

a. Check the entry for $n = 3$ in the table.
b. Use the remainder with $e^{0.5}$ to bound the error of $T_3(0.5)$, and compare with the true error (true value 1.64872).
c. Find the smallest n that guarantees an error below 10^{-6}.
d. Explain why the same polynomial would be less accurate at $x = 3$.

Question 16

Question 16 – A roller coaster profile *(Section C · Problem solving)*

Part of a roller coaster track has height $h(x) = 0.01x^3 - 0.3x^2 + 2.1x + 5$ metres, where x is the horizontal distance in metres and $0 \le x \le 20$.

a. Find $h'(x)$ and $h''(x)$.
b. Find the x-coordinates of the local maximum and local minimum, to 2 decimal places, and the heights there.
c. Find where the track descends most steeply and the gradient there.
d. Compare the heights at the ends $x = 0$ and $x = 20$ with the stationary values. What are the highest and lowest points on the track?

Question 17

Question 17 – Compound interest and the number e *(Section C · Problem solving)*

A bank pays 6% per year, compounded n times per year, so £5 000 grows to

$5000\left(1+\dfrac{0.06}{n}\right)^{10n}$ after 10 years.

Compounding	Value after 10 years
Yearly ($n=1$)	£8 954.24
Monthly ($n=12$)	£9 096.98
Daily ($n=365$)	£9 110.14
Continuously	?

a. Put $h=\dfrac{1}{n}$ and write $\ln\left(1+\dfrac{0.06}{n}\right)^{n}=\dfrac{\ln(1+0.06h)}{h}$.
b. Use l'Hôpital's rule as $h\to 0$ to find the limit of this expression.
c. Deduce that the continuous value is $5000e^{0.6}$ and calculate it.
d. By how much does daily compounding fall short of continuous compounding?

Question 18

Question 18 – Newton's method *(Section C · Problem solving)*

Newton's method finds roots using the iteration $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$. It is used by calculators to find roots of $f(x)=x^3+x-1$.

a. Show that f has exactly one real root by studying f' and the values $f(0)$ and $f(1)$.
b. Write the iteration for this function.
c. Starting from $x_0=0.7$, find x_1, x_2 and x_3 to 6 decimal places.
d. Explain why the sequence converges so quickly.

Question 19

Question 19 – A growing sand pile *(Section C · Problem solving)*

Sand falls from a conveyor belt at 2 m^3 per minute and forms a cone whose height always equals its radius. The volume of a cone is $V=\dfrac{1}{3}\pi r^2 h$.

Time (min)	2	6	10
Height (m)	?	?	?

a. Show that $V=\dfrac{1}{3}\pi h^3$ and that $\dfrac{dh}{dt}=\dfrac{2}{\pi h^2}$.
b. Find how fast the height increases when $h=3$.
c. Find V when $h=3$ and the time it takes to reach that height.
d. Fill in the table of heights, to 2 decimal places, and comment on the pattern of

$\dfrac{dh}{dt}$.

Question 20

Question 20 – Challenge: bounding the logarithm *(Section D · Challenge)*

Prove that $x - \dfrac{x^2}{2}$ for every $x > 0$, in the following steps.

a. Let $g(x) = x - \ln(1+x)$. Show that $g'(x) > 0$ for $x > 0$ and $g(0) = 0$. Deduce the right-hand inequality.

b. Let $f(x) = \ln(1+x) - x + \dfrac{x^2}{2}$. Show that $f'(x) = \dfrac{x^2}{1+x}$ and deduce the left-hand inequality.

c. Use the inequality with $x = 0.1$ to give lower and upper bounds for $\ln 1.1$.

d. The true value is 0.09531. How close is the lower bound?