Answers - Differentiation and its applications

Year: University Year 1

Answer to Question 1

a. Product rule with $u = x^3$, $v = \ln x$: $f'(x) = 3x^2 \ln x + x^3 \cdot \frac{1}{x} = x^2(3\ln x + 1)$.

b. Product rule: $g'(x) = 2e^{2x}\sin x + e^{2x}\cos x = e^{2x}(2\sin x + \cos x)$.

c. Quotient rule: $h'(x) = \frac{2x(x-1) - (x^2+1)\cdot 1}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2} = \frac{x^2 - 2x - 1}{(x-1)^2}$.

d. Chain rule with $u = 1 + x^2$: $k'(x) = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$.

Answer to Question 2

a. Take logarithms: $\ln y = x \ln x$. Differentiate: $\frac{1}{y}\frac{dy}{dx} = \ln x + 1$. So $\frac{dy}{dx} = x^x(\ln x + 1)$.

b. Differentiate implicitly: $2x + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{y}$. At $(3, 4)$ it is $-\frac{3}{4}$.

c. Differentiate: $3x^2 + 3y^2y' = 9y + 9xy'$. Collect: $y'(3y^2 - 9x) = 9y - 3x^2$. At $(2, 4)$: $y'(48 - 18) = 36 - 12$, so $30y' = 24$ and $y' = \frac{4}{5}$. (Check the point: $8 + 64 = 72 = 9 \times 2 \times 4$. Yes.)

Answer to Question 3

a. $\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2$.

b. $\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x - (x+h)}{hx(x+h)} = \frac{-1}{x(x+h)} \to -\frac{1}{x^2}$.

c. Multiply top and bottom by the conjugate: $\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \to \frac{1}{2\sqrt{x}}$.

Answer to Question 4

a. $f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. Since $f''(x) = 6x$: $f''(1) = 6 > 0$, a local **minimum** with $f(1) = -2$; $f''(-1) = -6$, a local **maximum** with $f(-1) = 2$.

b. $g'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) = 0$ gives $x = 0$ or $x = 3$. Now $g''(x) = 12x^2 - 24x$. At $x = 3$: $g'' = 108 - 72 = 36 > 0$, a **minimum** with $g(3) = -27$. At $x = 0$: $g''(0) = 0$, so the test fails. Check the sign of g': it is negative on both sides of 0 (since $4x^2 \geq 0$ and $x - 3$), so $x = 0$ is a **stationary point of inflection**, with $g(0) = 0$.

Answer to Question 5

a. The form is $\frac{0}{0}$. Differentiate top and bottom: $\frac{e^x - 1}{2x}$, still $\frac{0}{0}$. Again: $\frac{e^x}{2} \to \frac{1}{2}$.

b. The form is $0 \times (-\infty)$. Rewrite as $\frac{\ln x}{1/x}$, of the form $\frac{-\infty}{\infty}$. Differentiate: $\frac{1/x}{-1/x^2} = -x \to \mathbf{0}$.

c. The form is $\frac{\infty}{\infty}$. Apply the rule twice: $\frac{2x}{e^x} \to \frac{2}{e^x} \to \mathbf{0}$.

Answer to Question 6

a. $f = \cos x$, $f' = -\sin x$, $f'' = -\cos x$, $f''' = \sin x$, $f^{(4)} = \cos x$; at 0 these are $1, 0, -1, 0, 1$. So $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

b. $f(0) = 0$, $f' = \frac{1}{1+x}$ gives 1, $f'' = -\frac{1}{(1+x)^2}$ gives -1, $f''' = \frac{2}{(1+x)^3}$ gives 2. So $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$.

c. $e^u = 1 + u + \frac{u^2}{2} + \cdots$ with $u = -x^2$: $e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2}$.

Answer to Question 7

a. The base has sides $30 - 2x$ and $20 - 2x$ and the height is x. Expanding, $x(600 - 100x + 4x^2) = 4x^3 - 100x^2 + 600x$.

b. Both base lengths must be positive, so 0.

c. $V' = 12x^2 - 200x + 600 = 0$, so $3x^2 - 50x + 150 = 0$ and $x = \frac{50 \pm \sqrt{2500 - 1800}}{6} = \frac{50 \pm \sqrt{700}}{6}$. Only $x = \frac{50 - \sqrt{700}}{6} \approx$ **3.92 cm** lies in

$(0, 10)$. Since $V'' = 24x - 200 = -105.8$ there, it is a **maximum**.

d. $V(3.92) \approx$ **1 056 cm**^{3}.

Answer to Question 8

a. $\pi r^2 h = 330$, so $h = \dfrac{330}{\pi r^2}$. Then $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{660}{r}$.

b. $S' = 4\pi r - \dfrac{660}{r^2} = 0$ gives $r^3 = \dfrac{165}{\pi}$, so $r = \left(\dfrac{165}{\pi}\right)^{1/3} \approx$ **3.74 cm**.

$S'' = 4\pi + \dfrac{1320}{r^3} > 0$ so this is a minimum.

c. $h = \dfrac{330}{\pi r^2}$ and $\pi r^3 = 165$ means $\pi r^2 = \dfrac{165}{r}$, so $h = \dfrac{330r}{165} = 2r \approx$ **7.49 cm**, the diameter.

d. $S \approx 2\pi(14.02) + \dfrac{660}{3.745} \approx 88.1 + 176.2 =$ **264 cm**^{2}. Real cans are taller and thinner because they are easier to hold and stack, and because the lid and seams cost more than the sides.

Answer to Question 9

a. $R = (100 - 0.02x)x = 100x - 0.02x^2$. Then $P = R - C = 100x - 0.02x^2 - 2000 - 20x - 0.005x^2 = 80x - 0.025x^2 - 2000$.

b. $P' = 80 - 0.05x = 0$ gives $x =$ **1 600 cases**. $P'' = -0.05$, so it is a maximum.

c. $P(1600) = 128\,000 - 64\,000 - 2000 =$ **£62 000**. The price is $100 - 0.02 \times 1600 =$ **£68**, matching the table.

d. Beyond 1 600 cases the extra cost of making a case plus the fall in price on all the other cases is greater than the extra income, so P' and profit falls.

Answer to Question 10

a. Distance 10 miles, time $\dfrac{8}{60}$ h, so the average speed is $10 \div \dfrac{2}{15} =$ **75 mph**.

b. The MVT gives $c \in (0, 8)$ with $s'(c) = \dfrac{s(8) - s(0)}{8 - 0} = \dfrac{10}{8} = 1.25$ miles per minute, that is **75 mph**. So at time c the instantaneous speed was 75 mph, above the limit.

c. The position function s must be **continuous on** $[0, 8]$ and **differentiable on** $(0, 8)$. A real car satisfies both.

d. No. Without continuity the car could, in theory, jump from one place to another (impossible physically), and the theorem would not apply. The physical fact that cars move continuously is what makes the argument valid.

Answer to Question 11

a. By Pythagoras the ladder, wall and ground form a right-angled triangle with hypotenuse 5. Differentiating with respect to t: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.

b. When $x = 3$, $y = 4$. Then $3(0.5) + 4\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -\dfrac{1.5}{4} = -0.375$ **m/s** (the top moves down).

c. When $x = 4$, $y = 3$. Then $4(0.5) + 3\dfrac{dy}{dt} = 0$, so $\dfrac{dy}{dt} = -\dfrac{2}{3} \approx -0.667$ **m/s**: speed 0.667 m/s.

d. As $x \to 5$, $y \to 0$ and $\dfrac{dy}{dt} = -\dfrac{0.5x}{y} \to -\infty$. The top would move infinitely fast, which is impossible: in practice the ladder leaves the wall before the end.

Answer to Question 12

a. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 100$, so $\dfrac{dr}{dt} = \dfrac{100}{4\pi r^2} = \dfrac{25}{\pi r^2}$.

b. At $r = 10$: $\dfrac{25}{100\pi} = \dfrac{1}{4\pi} \approx$ **0.0796 cm/s**.

c. $\dfrac{dS}{dt} = 8\pi r\dfrac{dr}{dt} = 8\pi r \cdot \dfrac{25}{\pi r^2} = \dfrac{200}{r}$. At $r = 10$ this is **20 cm^2/s**.

d. As r grows the radius and the surface area both increase more slowly, because the same extra volume is spread over a bigger surface.

Answer to Question 13

a. $F'(v) = 0.004v - 0.3 = 0$ gives $v =$ **75 km/h**; $F'' = 0.004 > 0$ so it is a minimum. $F(75) = 0.002(5625) - 22.5 + 15 =$ **3.75 L/100 km**.

b. $3 \times 3.75 = 11.25$ L, costing $11.25 \times 1.50 \approx$ **£16.88**.

c. $F(120) = 28.8 - 36 + 15 = 7.8$ L/100 km. So $3 \times 7.8 = 23.4$ L, costing $23.4 \times 1.50 =$ **£35.10**.

d. Saving $= 35.10 - 16.88 \approx$ **£18.22**. Times: $300 \div 75 = 4$ hours and $300 \div 120 = 2.5$ hours. The faster journey saves 1.5 hours but costs more than twice as much.

Answer to Question 14

a. $v = 40 - 9.8t$ m/s and $a = -9.8$ **m/s^2** (constant, due to gravity).

b. The highest point is where $v = 0$: $t = \dfrac{40}{9.8} \approx$ **4.08 s**. Then

$h = 40(4.082) - 4.9(4.082)^2 \approx$ **81.6 m**. (Check: $\dfrac{40^2}{2 \times 9.8} = 81.6$.)

c. $h = 0$ gives $t(40 - 4.9t) = 0$, so $t = \dfrac{40}{4.9} \approx$ **8.16 s**. Then $v = 40 - 9.8(8.163) = -40$, a speed of **40 m/s**, the same as the launch speed.

d. Rolle's theorem (the special case of the mean value theorem) applies on $[0, 8.16]$ because $h(0) = h(8.16) = 0$ and h is a polynomial. So $h'(c) = 0$ for some c, and indeed $c = 4.08$ s, exactly halfway.

Answer to Question 15

a. $T_3(0.5) = 1 + 0.5 + \dfrac{0.25}{2} + \dfrac{0.125}{6} = 1 + 0.5 + 0.125 + 0.020833 =$ **1.645833**.

b. $|R_3| \le \dfrac{1.7 \times 0.5^4}{4!} = \dfrac{1.7 \times 0.0625}{24} \approx$ **0.0044**. The true error is $1.648721 - 1.645833 \approx 0.0029$, which is within the bound.

c. Need $\dfrac{1.7 \times 0.5^{n+1}}{(n+1)!}$. For $n = 6$ the bound is $1.7 \times \dfrac{0.5^7}{5040} \approx 2.6 \times 10^{-6}$ (too big). For $n = 7$ it is $1.7 \times \dfrac{0.5^8}{40320} \approx 1.6 \times 10^{-7}$. So the smallest n is **7**.

d. At $x = 3$ the powers 3^{n+1} are large, and the factor e^{ξ} could be as big as $e^3 \approx 20$, so many more terms are needed for the same accuracy.

Answer to Question 16

a. $h'(x) = 0.03x^2 - 0.6x + 2.1$ and $h''(x) = 0.06x - 0.6$.

b. $h' = 0$ gives $x^2 - 20x + 70 = 0$, so $x = 10 \pm \sqrt{30} \approx 4.52$ and 15.48. Since $h''(4.52) = -0.33$ it is a local **maximum**, with $h \approx$ **9.29 m**. Since $h''(15.48) = 0.33 > 0$ it is a local **minimum**, with $h \approx$ **2.71 m**.

c. The gradient h' is smallest where $h'' = 0$, that is $x = 10$. There $h'(10) = 3 - 6 + 2.1 = -0.9$, so the track drops 0.9 m per metre horizontally (about 42°).

d. $h(0) = 5$ and $h(20) = 7$. Comparing all four values 5, 9.29, 2.71, 7: the highest point is **9.29 m** (at $x \approx 4.52$) and the lowest is **2.71 m** (at $x \approx 15.48$).

Answer to Question 17

a. $n \ln\left(1 + \dfrac{0.06}{n}\right) = \dfrac{1}{h}\ln(1 + 0.06h)$.

b. As $h \to 0$ the form is $\dfrac{0}{0}$. Differentiating, $\dfrac{0.06/(1 + 0.06h)}{1} \to$ **0.06**. So

$\left(1+\dfrac{0.06}{n}\right)^{n} \to e^{0.06}$.

c. After 10 years the factor is $\left(e^{0.06}\right)^{10} = e^{0.6}$. So the value is $5000e^{0.6} \approx$ **£9 110.59**.

d. $9110.59 - 9110.14 =$ **£0.45**. Daily compounding is already very close to the continuous limit.

Answer to Question 18

a. $f'(x) = 3x^2 + 1 > 0$ for all x, so f is strictly increasing and crosses zero at most once. $f(0) = -1$ and $f(1) = 1 > 0$, so by the intermediate value theorem there is exactly **one real root**, in $(0, 1)$.

b. $x_{n+1} = x_n - \dfrac{x_n^3 + x_n - 1}{3x_n^2 + 1}$.

c. $x_1 = 0.7 - \dfrac{0.343 + 0.7 - 1}{1.47 + 1} = 0.7 - \dfrac{0.043}{2.47} \approx$ **0.682591**; $x_2 \approx$ **0.682328**; $x_3 \approx$ **0.682328**. The root is about 0.682328.

d. Near a root where $f' \neq 0$, the error is roughly squared at each step (quadratic convergence), so the number of correct digits about doubles each time.

Answer to Question 19

a. With $r = h$, $V = \dfrac{1}{3}\pi h^2 \cdot h = \dfrac{1}{3}\pi h^3$. Then $\dfrac{dV}{dt} = \pi h^2 \dfrac{dh}{dt} = 2$, so $\dfrac{dh}{dt} = \dfrac{2}{\pi h^2}$.

b. At $h = 3$: $\dfrac{2}{9\pi} \approx$ **0.0707 m/min**.

c. $V = \dfrac{1}{3}\pi \cdot 27 = 9\pi \approx 28.27$ m^3. At 2 m^3/min this takes $\dfrac{9\pi}{2} \approx$ **14.1 minutes**.

d. After time t, $V = 2t$, so $h = \left(\dfrac{6t}{\pi}\right)^{1/3}$. For $t = 2$: $h = (3.82)^{1/3} \approx$ **1.56**; $t = 6$: $(11.46)^{1/3} \approx$ **2.25**; $t = 10$: $(19.10)^{1/3} \approx$ **2.67**. The height keeps rising, but ever more slowly, since $\dfrac{dh}{dt}$ is proportional to $\dfrac{1}{h^2}$.

Answer to Question 20

a. $g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$. So g is strictly increasing and $g(0) = 0$, which gives $g(x) > 0$, that is $\ln(1+x)$.

b. $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{1 - (1+x) + x(1+x)}{1+x} = \dfrac{x^2}{1+x} > 0$ for $x > 0$. With

$f(0) = 0$, f is strictly increasing, so $f(x) > 0$ for $x > 0$: $\ln(1 + x) > x - \frac{x^2}{2}$.

c. With $x = 0.1$: $0.1 - 0.005$, that is 0.095.
d. The lower bound 0.095 differs from 0.09531 by about 0.0003, a relative error of 0.3 per cent.