Test answers - Differential equations

Year: University Year 1 · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

The equation is separable, so collect the y terms on the left and the x terms on the right: $\int y\,dy = \int x\,dx$ (1). Integrating both sides gives $\frac{y^2}{2} = \frac{x^2}{2} + c$ (1). Using the initial condition $y(0) = 2$: $2 = 0 + c$, so $c = 2$ and $y^2 = x^2 + 4$ (1). Since we need $y > 0$, we take the positive square root: $y = \sqrt{x^2 + 4}$ (1).

Check: differentiating $y^2 = x^2 + 4$ implicitly gives $2yy' = 2x$, so $y' = \frac{x}{y}$, and $y(0) = \sqrt{4} = 2$. The negative root is rejected because it fails the initial condition. The solution curve is the upper branch of the hyperbola $y^2 - x^2 = 4$.

Answer to Question 2 (4 marks)

The equation is linear with $P = 1$, so the integrating factor is $\mu = e^{\int 1\,dx} = e^x$. Multiplying through gives $(e^x y)' = 2e^{2x}$ (1). Integrating both sides, $e^x y = e^{2x} + C$ (1), so the general solution is $y = e^x + Ce^{-x}$. The condition $y(0) = 2$ gives $1 + C = 2$, so $C = 1$ (1). Therefore $y = e^x + e^{-x} = 2\cosh x$ (1). Check: $y' + y = (e^x - e^{-x}) + (e^x + e^{-x}) = 2e^x$, as required.

A common error is to forget to multiply the right-hand side by the integrating factor, which would give $(e^x y)' = 2e^x$ and the wrong solution $y = 2 + Ce^{-x}$. The constant C must be added before dividing by e^x.

Answer to Question 3 (4 marks)

The auxiliary equation is $m^2 - 4m + 4 = (m - 2)^2 = 0$, which has the repeated root 2, so the general solution is $y = (A + Bx)e^{2x}$ (1). Putting $x = 0$ gives $y(0) = A = 1$ (1). Differentiating with the product rule, $y' = (B + 2A + 2Bx)e^{2x}$, so $y'(0) = B + 2A = B + 2 = 3$, which gives $B = 1$ (1). Hence the solution is $y = (1 + x)e^{2x}$ (1). Check: $y(0) = 1$ and $y' = (3 + 2x)e^{2x}$ gives $y'(0) = 3$, so both conditions hold.

The repeated root is the reason for the factor x: writing $y = Ae^{2x} + Be^{2x}$ gives only one independent solution and cannot satisfy both conditions.

Answer to Question 4 (4 marks)

a. Let $u = T - 20$, the excess temperature. Then $u' = -ku$, so $u = u_0 e^{-kt}$ with $u_0 = 200 - 20 = 180$, which gives $T = 20 + 180e^{-kt}$ (1). At $t = 10$:

$110 = 20 + 180e^{-10k}$, so $e^{-10k} = \frac{1}{2}$ (1). Taking logarithms, $k = \frac{\ln 2}{10} \approx 0.0693$ per minute (1).

b. Set $50 = 20 + 180e^{-kt}$, so $e^{-kt} = \frac{1}{6}$ and $t = \frac{\ln 6}{k} = \frac{10 \ln 6}{\ln 2} \approx$ **25.8 minutes** (1).

The bar halves its excess temperature every 10 minutes, so this answer is sensible: it lies between 20 minutes (excess 45) and 30 minutes (excess 22.5).

Answer to Question 5 (4 marks)

First solve the homogeneous equation. The auxiliary equation is $m^2 + m - 2 = (m + 2)(m - 1) = 0$, with roots 1 and -2, so the complementary function is $y_c = Ae^x + Be^{-2x}$ (2 marks). Since 2 is not a root, try a particular integral $y_p = ke^{2x}$. Then $y_p' = 2ke^{2x}$, $y_p'' = 4ke^{2x}$, and substituting gives $(4 + 2 - 2)k = 6$, so $k = \frac{3}{2}$ (1). The general solution is the sum: $y = Ae^x + Be^{-2x} + \frac{3}{2}e^{2x}$ (1).

Check the particular integral: $\frac{3}{2}(4 + 2 - 2)e^{2x} = 6e^{2x}$. If the right-hand side had been e^x, the trial function would need to be kxe^x because 1 is a root of the auxiliary equation.