Worksheet - Differential equations

Year: University Year 1 · 20

Show your working.

Question 1

Question 1 – Separable equations *(Section A · Skills practice)*

Solve each differential equation by separating the variables. Give y explicitly where you can, and use the initial condition when one is given.

a. $\dfrac{dy}{dx} = 3x^2y$ [3 marks]

b. $\dfrac{dy}{dx} = e^{x-y}$ [3 marks]

c. $x\dfrac{dy}{dx} = y$, with $y(1) = 3$ [3 marks]

d. $(1+x^2)\dfrac{dy}{dx} = xy$, with $y(0) = 2$ [4 marks]

Question 2

Question 2 – Integrating factors *(Section A · Skills practice)*

Solve each first-order linear equation using an integrating factor.

a. $y' + 3y = 6$ [3 marks]

b. $x\dfrac{dy}{dx} + 2y = x^2$ for $x > 0$ [4 marks]

c. $y' + y = e^{-x}$, with $y(0) = 2$ [4 marks]

Question 3

Question 3 – Auxiliary equations *(Section A · Skills practice)*

Find the general solution of each equation. State whether the roots of the auxiliary equation are real and distinct, repeated, or complex.

a. $y'' - 7y' + 12y = 0$ [3 marks]

b. $y'' + 6y' + 9y = 0$ [3 marks]

c. $y'' + 2y' + 10y = 0$ [3 marks]

Question 4

Question 4 – Initial value problems *(Section A · Skills practice)*

Solve each initial value problem.

a. $y'' - 4y = 0$, with $y(0) = 0$ and $y'(0) = 4$ [4 marks]
b. $y'' + 4y' + 13y = 0$, with $y(0) = 2$ and $y'(0) = -1$ [5 marks]

Question 5

Question 5 – Particular integrals *(Section A · Skills practice)*

Find a particular integral for each equation, and hence write down the general solution of (a).

a. $y'' - y' - 6y = 12$ [3 marks]
b. $y'' + 9y = 18\cos 2x$ [3 marks]
c. $y'' - 3y' + 2y = e^{x}$ [4 marks]

Question 6

Question 6 – Classifying and checking *(Section A · Skills practice)*

(a) State the order of each equation and whether it is linear: (i) $y''' + xy = 0$, (ii) $y' = y^2$, (iii) $y'' + \sin y = 0$. (b) Show that $y = xe^{2x}$ solves $y'' - 4y' + 4y = 0$. (c) Find A and B so that $y = Ae^{x} + Be^{-x}$ satisfies $y(0) = 3$ and $y'(0) = 1$. [8 marks in total]

Question 7

Question 7 – Growing bacteria *(Section B · Real-life practice)*

A culture in a university laboratory contains 500 bacteria. Their number P grows at a rate proportional to P, so $\dfrac{dP}{dt} = rP$, where t is the time in hours. After 3 hours there are 1 500 bacteria.

a. Solve the equation to show that $P = 500e^{rt}$, and find r to three significant figures.
b. Find the doubling time.
c. How many bacteria are there after 8 hours?
d. The lab must stop the experiment when there are 1 000 000 bacteria. After how many hours is that?

Question 8

Question 8 – A cooling cup of coffee *(Section B · Real-life practice)*

Newton's law of cooling says that the temperature T (in °C) of an object in a room at temperature T_s satisfies $\dfrac{dT}{dt} = -k(T - T_s)$. A cup of coffee at 85°C is left in a room at 21°C. After 5 minutes it has cooled to 68°C.

a. Show that $T = 21 + 64e^{-kt}$.

b. Find k.
c. Find the temperature after 10 minutes.
d. It is safe to drink at 40°C. When is that?

Question 9

Question 9 – Salt in a mixing tank *(Section B · Real-life practice)*

A tank holds 200 litres of pure water. Brine containing 0.4 kg of salt per litre flows in at 5 litres per minute. The well-stirred mixture flows out at 5 litres per minute. Let Q be the mass of salt in kg after t minutes.

a. Explain why $\dfrac{dQ}{dt} = 2 - \dfrac{Q}{40}$.
b. Solve the equation with $Q(0) = 0$.
c. How long does it take for the tank to hold 50 kg of salt?
d. What happens to Q in the long run?

Question 10

Question 10 – Charging a capacitor *(Section B · Real-life practice)*

In an electrical circuit a battery of voltage $E = 12$ V charges a capacitor through a resistor. The charge Q (in coulombs) satisfies $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = E$. Here $R = 4000\ \Omega$ and $C = 250\ \mu\text{F} = 2.5 \times 10^{-4}$ F, so $RC = 1$ second. At $t = 0$ the capacitor is empty.

a. Show that $Q = CE\left(1 - e^{-t}\right)$.
b. Write down the voltage across the capacitor, $V = Q/C$, and the current $I = \dfrac{dQ}{dt}$.
c. How long does it take for the voltage to reach 9 V?
d. What is the current at $t = 0$?

Question 11

Question 11 – A mass on a spring *(Section B · Real-life practice)*

A block of mass 0.5 kg hangs on a spring of stiffness $k = 32$ N/m. It is pulled 0.1 m below its rest position and released from rest. The displacement x metres satisfies $0.5x'' = -32x$ if there is no resistance.

a. Show that $x'' + 64x = 0$, and solve it.
b. Find the period and the maximum speed.
c. A damper is now added, giving $0.5x'' + 2x' + 32x = 0$. Find m in the auxiliary equation and describe the motion.
d. By what factor does the amplitude shrink each second in the damped motion?

Question 12

Question 12 – Medicine in the bloodstream *(Section B · Real-life practice)*

A drug is given through a drip at a constant rate of 4 mg per hour. The body removes the drug at a rate proportional to the amount C (in mg) in the blood, with constant 0.2 per hour. So $\frac{dC}{dt} = 4 - 0.2C$, and $C(0) = 0$.

a. Solve the equation.
b. Find the steady level that the amount approaches.
c. When does the amount reach 15 mg?
d. The doctor wants 15 mg after only 2 hours by giving an initial dose $C(0) = C_0$ with the same drip. Find C_0 to 2 decimal places.

Question 13

Question 13 – Savings with continuous interest *(Section B · Real-life practice)*

An account earns interest continuously at 4% per year, and its owner also pays in £2 400 per year, spread evenly through the year. The balance B (in £) after t years satisfies $\frac{dB}{dt} = 0.04B + 2400$. The opening balance is £1 000.

a. Solve the equation for B.
b. Find the balance after 10 years.
c. How long until the balance reaches £100 000?

Question 14

Question 14 – Falling with air resistance *(Section B · Real-life practice)*

A skydiver of mass 70 kg falls with air resistance proportional to speed. Taking $g = 9.8$ m/s^2 and a drag constant of 14 kg/s, Newton's second law gives $70\frac{dv}{dt} = 70 \times 9.8 - 14v$, so $\frac{dv}{dt} = 9.8 - 0.2v$. She starts from rest.

a. Solve for $v(t)$.
b. Find the terminal velocity.
c. When does she reach 40 m/s?
d. Find the distance fallen in the first 10 seconds.

Question 15

Question 15 – Fish in a lake *(Section C · Problem solving)*

The number of fish P in a lake grows according to the logistic equation

$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{2000}\right)$, where t is in years. At $t = 0$ there are 200 fish.

a. Use the substitution $u = \frac{1}{P}$ or partial fractions to show that $P = \frac{2000}{1 + 9e^{-0.3t}}$.
b. When does the population reach 1 000?
c. The lake is then fished at a constant rate of 100 fish per year, so $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{2000}\right) - 100$. Find the two equilibrium populations.
d. Which equilibrium is stable? What happens if the population falls below the other one?

Question 16

Question 16 – Designing a car suspension *(Section C · Problem solving)*

A quarter of a car of mass $m = 400$ kg is supported by a spring of stiffness $k = 40\,000$ N/m and a damper with damping constant c (N s/m). The vertical displacement x metres from equilibrium satisfies $400x'' + cx' + 40\,000x = 0$.

a. Show that the undamped angular frequency is 10 rad/s.
b. Find the critical value of c, that is the value for which the auxiliary equation has a repeated root.
c. For $c = 2000$ show that the roots are complex, and write the general solution.
d. For $c = 12\,000$ find the roots. Explain why the car does not oscillate.
e. Which type of damping gives the fastest return to equilibrium without bouncing? Explain.

Question 17

Question 17 – Resonance on a footbridge *(Section C · Problem solving)*

Engineers model the sideways sway x (in cm) of a footbridge by $x'' + 4x = 3\cos \omega t$, where the forcing comes from the footsteps of walkers. The bridge starts at rest, so $x(0) = 0$ and $x'(0) = 0$.

a. Find the complementary function.
b. For $\omega = 1$ find a particular integral and then the solution satisfying the conditions.
c. For $\omega = 2$ explain why the usual guess fails, find a particular integral of the form $kt\sin 2t$, and solve the problem.
d. Describe what happens to the sway in part (c) as time passes. Why do engineers avoid this situation?

Question 18

Question 18 – A rumour at the university *(Section C · Problem solving)*

There are 1 000 students on a campus. Let I be the number who have heard a rumour

after t days. Each student who knows the rumour tells others at a rate proportional to the number who do not know it, so $\dfrac{dI}{dt} = 0.0004\,I(1000 - I)$. Five students start the rumour: $I(0) = 5$.

a. Show that $\dfrac{dI}{dt} = 0.4I\left(1 - \dfrac{I}{1000}\right)$.

b. Solve the equation and show that $I = \dfrac{1000}{1 + 199e^{-0.4t}}$.

c. When have 500 students heard the rumour?

d. At what rate is the rumour spreading at that moment? Explain why this is the fastest rate.

Question 19

Question 19 – Emptying a tank *(Section C · Problem solving)*

A cylindrical tank has cross-sectional area 2 m^2. Water leaves through a small hole of area 0.002 m^2 at the bottom. Torricelli's law gives the outflow speed as $\sqrt{2gh}$, so the height h (in m) satisfies $2\dfrac{dh}{dt} = -0.002\sqrt{19.6\,h}$ (using $g = 9.8$). Initially $h = 4$.

a. Show that $\dfrac{dh}{dt} = -0.001\sqrt{19.6}\sqrt{h}$.

b. Solve the equation to find $\sqrt{h}$ as a function of t.

c. How long, to the nearest minute, does the tank take to empty?

d. How long does it take to drop from 4 m to 1 m?

Question 20

Question 20 – Challenge: a homogeneous equation *(Section D · Challenge)*

Consider $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$ for $x > 0$, $y > 0$, with $y(1) = 2$.

a. Put $y = vx$ and show that $x\dfrac{dv}{dx} = \dfrac{1}{v}$.

b. Solve for v, and hence find y in terms of x.

c. Find $y(e)$ exactly.

d. Check your answer by differentiating.