Answers - Differential equations

Year: University Year 1

Answer to Question 1

a. Separate: $\int \frac{dy}{y} = \int 3x^2\,dx$, so $\ln|y| = x^3 + c$ and $y = Ce^{x^3}$ with $C = \pm e^c$.

b. Write $e^{x-y} = e^x e^{-y}$ and separate: $\int e^y\,dy = \int e^x\,dx$. So $e^y = e^x + C$ and $y = \ln(e^x + C)$.

c. Separate: $\int \frac{dy}{y} = \int \frac{dx}{x}$, so $\ln y = \ln x + c$ and $y = Ax$. Then $y(1) = 3$ gives $A = 3$, so $y = 3x$.

d. Separate: $\int \frac{dy}{y} = \int \frac{x}{1+x^2}dx$, so $\ln y = \frac{1}{2}\ln(1+x^2) + c$ and $y = A\sqrt{1+x^2}$. Then $y(0) = 2$ gives $A = 2$, so $y = 2\sqrt{1+x^2}$.

Answer to Question 2

a. The equation is already in standard form $y' + P(x)y = Q(x)$ with $P = 3$. The integrating factor is $\mu = e^{\int 3\,dx} = e^{3x}$. Multiplying through, the left-hand side becomes an exact derivative: $(e^{3x}y)' = 6e^{3x}$. Integrating both sides, $e^{3x}y = 2e^{3x} + C$. Dividing by e^{3x}, the general solution is $y = 2 + Ce^{-3x}$.

b. First divide by x (allowed since $x > 0$) to reach standard form: $y' + \frac{2}{x}y = x$. Then $\mu = e^{\int 2/x\,dx} = e^{2\ln x} = x^2$. Multiplying, $(x^2y)' = x^3$. Integrating, $x^2y = \frac{x^4}{4} + C$, so the general solution is $y = \frac{x^2}{4} + \frac{C}{x^2}$.

c. Here $P = 1$, so $\mu = e^x$. Multiplying, $(e^x y)' = e^x \cdot e^{-x} = 1$, and integrating gives $e^x y = x + C$. Hence $y = (x + C)e^{-x}$. The condition $y(0) = 2$ gives $C = 2$, so the particular solution is $y = (x + 2)e^{-x}$.

Answer to Question 3

a. Trying $y = e^{mx}$ gives the auxiliary equation $m^2 - 7m + 12 = 0$. It factorises as $(m-3)(m-4) = 0$, so the roots 3 and 4 are real and distinct. Each root gives one exponential solution, and the general solution is $y = Ae^{3x} + Be^{4x}$.

b. The auxiliary equation is $m^2 + 6m + 9 = (m+3)^2 = 0$. There is one repeated root -3. A repeated root needs a second independent solution, obtained by multiplying by x, so $y = (A + Bx)e^{-3x}$.

c. The auxiliary equation $m^2 + 2m + 10 = 0$ has discriminant $4 - 40 = -36 < 0$, so

the roots are complex: $m = \dfrac{-2 \pm 6i}{2} = -1 \pm 3i$. With real part -1 and imaginary part 3, the general solution is $y = e^{-x}(A\cos 3x + B\sin 3x)$, a decaying oscillation.

Answer to Question 4

a. The auxiliary equation is $m^2 - 4 = 0$, with roots ± 2, so $y = Ae^{2x} + Be^{-2x}$. The first condition gives $y(0) = A + B = 0$. Differentiating, $y' = 2Ae^{2x} - 2Be^{-2x}$, so $y'(0) = 2A - 2B = 4$. Solving the two equations, $A = 1$ and $B = -1$. Therefore $y = e^{2x} - e^{-2x} = 2\sinh 2x$.

b. The auxiliary equation $m^2 + 4m + 13 = 0$ gives $m = -2 \pm 3i$, so $y = e^{-2x}(A\cos 3x + B\sin 3x)$. Putting $x = 0$: $y(0) = A = 2$. Using the product rule, $y' = e^{-2x}\left[(-2A + 3B)\cos 3x + (-2B - 3A)\sin 3x\right]$, so $y'(0) = -2A + 3B = -4 + 3B = -1$, which gives $B = 1$. Hence the solution is $y = e^{-2x}(2\cos 3x + \sin 3x)$.

Answer to Question 5

a. Try $y_p = k$: then $-6k = 12$ and $k = -2$. The complementary function is $m^2 - m - 6 = (m - 3)(m + 2)$, so $y = Ae^{3x} + Be^{-2x} - 2$.

b. Try $y_p = k\cos 2x$ (a sine term is not needed because there is no first derivative). Then $(-4 + 9)k = 18$ and $k = \dfrac{18}{5}$.

c. The function e^x already solves the homogeneous equation, because $m = 1$ is a root of $m^2 - 3m + 2 = 0$. So try $y_p = kxe^x$. Then $y_p'' - 3y_p' + 2y_p = k(2 + x)e^x - 3k(1 + x)e^x + 2kxe^x = -ke^x$. So $k = -1$ and $y_p = -xe^x$.

Answer to Question 6

a. The order is the highest derivative present. (i) Order 3; it is linear because y and its derivatives appear only to the first power with coefficients in x. (ii) Order 1; it is not linear because of the term y^2. (iii) Order 2; it is not linear because y appears inside $\sin y$.

b. Using the product rule, $y' = (1 + 2x)e^{2x}$ and $y'' = (4 + 4x)e^{2x}$. Substituting, $y'' - 4y' + 4y = [4 + 4x - 4 - 8x + 4x]e^{2x} = 0$, so $y = xe^{2x}$ is indeed a solution (it comes from the repeated root 2).

c. $y(0) = A + B = 3$ and, since $y' = Ae^x - Be^{-x}$, $y'(0) = A - B = 1$. Adding the equations, $2A = 4$, so $A = 2$ and then $B = 1$.

Answer to Question 7

a. Separating the variables, $\int \frac{dP}{P} = \int r\,dt$, so $\ln P = rt + c$ and $P = P_0 e^{rt}$. At $t = 0$ there are 500 bacteria, so $P = 500e^{rt}$. After 3 hours, $1500 = 500e^{3r}$, so $e^{3r} = 3$ and $r = \frac{\ln 3}{3} \approx$ **0.366 per hour**.

b. The population doubles when $e^{rt} = 2$, so $t = \frac{\ln 2}{r} = \frac{3\ln 2}{\ln 3} \approx$ **1.89 hours**. This does not depend on the starting number.

c. Since $e^r = 3^{1/3}$, $P(8) = 500 \times 3^{8/3} \approx$ **9 360 bacteria**.

d. Solve $500 \times 3^{t/3} = 10^6$, so $3^{t/3} = 2000$. Taking logarithms, $t = \frac{3\ln 2000}{\ln 3} \approx$ **20.8 hours**; the experiment must stop after about 20.8 hours.

Answer to Question 8

a. Let $u = T - 21$, the excess over room temperature. Then $u' = T' = -ku$, whose solution is $u = u_0 e^{-kt}$. Initially $u_0 = 85 - 21 = 64$, so $T = 21 + 64e^{-kt}$, as required.

b. At $t = 5$: $68 = 21 + 64e^{-5k}$, so $e^{-5k} = \frac{47}{64}$. Taking logarithms, $k = \frac{1}{5}\ln\frac{64}{47} \approx$ **0.0617 per minute**.

c. Since $e^{-10k} = \left(e^{-5k}\right)^2$, $T(10) = 21 + 64\left(\frac{47}{64}\right)^2 = 21 + \frac{47^2}{64} \approx$ **55.5 °C**.

d. Set $40 = 21 + 64e^{-kt}$, so $e^{-kt} = \frac{19}{64}$ and $t = \frac{1}{k}\ln\frac{64}{19} \approx$ **19.7 minutes**. The coffee is safe to drink about 20 minutes after it is poured.

Answer to Question 9

a. Salt enters at $5 \times 0.4 = 2$ kg per minute. The concentration in the tank is $\frac{Q}{200}$ kg per litre, so salt leaves at $5 \times \frac{Q}{200} = \frac{Q}{40}$ kg per minute. The net rate is $2 - \frac{Q}{40}$.

b. Rewrite as $Q' + \frac{Q}{40} = 2$. With $\mu = e^{t/40}$, $(e^{t/40}Q)' = 2e^{t/40}$, so $e^{t/40}Q = 80e^{t/40} + C$. Using $Q(0) = 0$ gives $C = -80$ and $Q = 80\left(1 - e^{-t/40}\right)$.

c. Solve $80(1 - e^{-t/40}) = 50$. Then $e^{-t/40} = \frac{3}{8}$ and $t = 40\ln\frac{8}{3} \approx$ **39.2 minutes**.

d. As $t \to \infty$, $e^{-t/40} \to 0$ so $Q \to$ **80 kg**. This equals 200×0.4: the tank ends with the same concentration as the incoming brine.

Answer to Question 10

a. Divide by R: $Q' + \dfrac{Q}{RC} = \dfrac{E}{R}$, that is $Q' + Q = \dfrac{E}{R}$ because $RC = 1$. With $\mu = e^t$: $(e^t Q)' = \dfrac{E}{R} e^t$, so $Q = \dfrac{E}{R} + Ke^{-t}$. Since $Q(0) = 0$, $K = -\dfrac{E}{R}$. As $\dfrac{E}{R} = \dfrac{12}{4000} = 3 \times 10^{-3} = CE$ (because $RC = 1$), $Q = CE(1 - e^{-t})$.

b. $V = \dfrac{Q}{C} = 12(1 - e^{-t})$ volts. The current is $I = \dfrac{dQ}{dt} = CEe^{-t} = 3 \times 10^{-3} e^{-t}$ amps.

c. Solve $12(1 - e^{-t}) = 9$: $e^{-t} = \dfrac{1}{4}$, so $t = \ln 4 \approx$ **1.39 seconds**.

d. $I(0) = 3 \times 10^{-3}$ A = **3 mA**.

Answer to Question 11

a. Divide $0.5x'' = -32x$ by 0.5 to get $x'' + 64x = 0$. The roots are $\pm 8i$, so $x = A\cos 8t + B\sin 8t$. From $x(0) = 0.1$, $A = 0.1$. From $x'(0) = 0$, $8B = 0$, so $B = 0$ and $x = 0.1\cos 8t$.

b. Period $= \dfrac{2\pi}{8} = \dfrac{\pi}{4} \approx$ **0.785 s**. Velocity $x' = -0.8\sin 8t$, so the maximum speed is **0.8 m/s**.

c. Multiply by 2: $x'' + 4x' + 64x = 0$, so $m^2 + 4m + 64 = 0$ and $m = -2 \pm \sqrt{60}\, i \approx -2 \pm 7.75i$. The roots are complex, so the motion is an oscillation inside a decaying envelope e^{-2t}: **underdamped**.

d. The envelope is e^{-2t}, so each second the amplitude is multiplied by $e^{-2} \approx$ **0.135**.

Answer to Question 12

a. Rewrite as a linear equation: $C' + 0.2C = 4$, with integrating factor $\mu = e^{0.2t}$. Then $(e^{0.2t}C)' = 4e^{0.2t}$, and integrating, $e^{0.2t}C = 20e^{0.2t} + K$. The condition $C(0) = 0$ gives $K = -20$, so $C = 20\left(1 - e^{-0.2t}\right)$.

b. As $t \to \infty$, $e^{-0.2t} \to 0$, so $C \to$ **20 mg**. This is the equilibrium where intake equals removal: $4 - 0.2C = 0$.

c. Solve $20(1 - e^{-0.2t}) = 15$: $e^{-0.2t} = \dfrac{1}{4}$, so $t = 5\ln 4 \approx$ **6.93 hours**.

d. With an initial dose, the same method gives $C = 20 + (C_0 - 20)e^{-0.2t}$. At $t = 2$: $15 = 20 + (C_0 - 20)e^{-0.4}$, so $C_0 - 20 = -5e^{0.4}$ and $C_0 = 20 - 5e^{0.4} \approx$ **12.54 mg**.

Answer to Question 13

a. Write the equation as $B' - 0.04B = 2400$. The integrating factor is $\mu = e^{-0.04t}$, so

$(e^{-0.04t}B)' = 2400e^{-0.04t}$. Integrating, $e^{-0.04t}B = -60\,000e^{-0.04t} + K$, hence $B = Ke^{0.04t} - 60\,000$. The opening balance $B(0) = 1000$ gives $K = 61\,000$, so $B = 61\,000e^{0.04t} - 60\,000$.

b. Substituting $t = 10$: $B(10) = 61\,000e^{0.4} - 60\,000 \approx$ **£31 001**.

c. Solve $61\,000e^{0.04t} - 60\,000 = 100\,000$, so $e^{0.04t} = \dfrac{160}{61}$. Taking logarithms, $t = 25\ln\dfrac{160}{61} \approx$ **24.1 years**. The balance first reaches £100 000 after about 24 years.

Answer to Question 14

a. Write $v' + 0.2v = 9.8$. The constant particular solution is $v = \dfrac{9.8}{0.2} = 49$, and the complementary function is $Ke^{-0.2t}$, so $v = 49 + Ke^{-0.2t}$. Starting from rest, $v(0) = 0$ gives $K = -49$, hence $v = 49\left(1 - e^{-0.2t}\right)$.

b. As $t \to \infty$, $e^{-0.2t} \to 0$, so the terminal velocity is **49 m/s**; at that speed drag balances weight.

c. Solve $49(1 - e^{-0.2t}) = 40$: $e^{-0.2t} = \dfrac{9}{49}$, so $t = 5\ln\dfrac{49}{9} \approx$ **8.47 seconds**.

d. Distance is the integral of speed:
$\displaystyle\int_0^{10} 49(1 - e^{-0.2t})\,dt = 49\left[t + 5e^{-0.2t}\right]_0^{10} = 49\left(10 + 5e^{-2} - 5\right) \approx$ **278 m**.

Answer to Question 15

a. Separate: $\displaystyle\int \frac{2000\,dP}{P(2000 - P)} = \int 0.3\,dt$. Partial fractions: $\dfrac{2000}{P(2000-P)} = \dfrac{1}{P} + \dfrac{1}{2000 - P}$, so [redacted]. Then $\dfrac{P}{2000 - P} = Ae^{0.3t}$ and $A = \dfrac{200}{1800} = \dfrac{1}{9}$. Solving, $P = \dfrac{2000}{1 + 9e^{-0.3t}}$.

b. $P = 1000$ when $9e^{-0.3t} = 1$, so $t = \dfrac{\ln 9}{0.3} \approx$ **7.32 years**.

c. Set $0.3P - 0.00015P^2 - 100 = 0$, that is $P^2 - 2000P + 666\,667 = 0$ (approximately, since $100/0.00015 = 666\,667$). So $P = 1000 \pm \sqrt{10^6 - 666\,667} \approx 1000 \pm 577$, giving $P \approx 423$ **and** $P \approx 1577$.

d. Let $f(P) = 0.3P(1 - P/2000) - 100$. Between the two roots $f > 0$, so the population rises towards 1 577 (**stable**). Below 423, f, so the population keeps falling: the fish **die out**. The lower equilibrium is unstable, so over-fishing can cause a collapse.

Answer to Question 16

a. With $c = 0$: $x'' + 100x = 0$, so $\omega^2 = 100$ and $\omega =$ **10 rad/s**.

b. The auxiliary equation $400m^2 + cm + 40\,000 = 0$ has a repeated root when $c^2 - 4(400)(40\,000) = 0$, so $c^2 = 6.4 \times 10^7$ and $c =$ **8 000 N s/m**.

c. With $c = 2000$: $m^2 + 5m + 100 = 0$, so $m = \dfrac{-5 \pm \sqrt{25 - 400}}{2} = -2.5 \pm \dfrac{\sqrt{375}}{2}i \approx -2.5 \pm 9.68i$. The roots are complex: $x = e^{-2.5t}(A\cos 9.68t + B\sin 9.68t)$.

d. With $c = 12\,000$: $m^2 + 30m + 100 = 0$, so $m = -15 \pm \sqrt{125} \approx -3.82$ **and** -26.18. Both roots are real and negative, so $x = Ae^{-3.82t} + Be^{-26.18t}$ decays without oscillating (overdamped).

e. **Critical damping** ($c = 8000$) returns to rest fastest without crossing equilibrium. Underdamping bounces, and overdamping creeps back slowly because of the small root -3.82.

Answer to Question 17

a. $m^2 + 4 = 0$ gives $m = \pm 2i$, so $x_c = A\cos 2t + B\sin 2t$.

b. Try $x_p = k\cos t$: $(-1 + 4)k = 3$, so $k = 1$. Then $x = A\cos 2t + B\sin 2t + \cos t$. From $x(0) = 0$, $A = -1$. From $x'(0) = 0$, $2B = 0$. So $x = \cos t - \cos 2t$.

c. Now $\cos 2t$ is already in x_c, so the guess $k\cos 2t$ would give zero on the left. Try $x_p = kt\sin 2t$. Then $x_p'' = 4k\cos 2t - 4kt\sin 2t$, so $x_p'' + 4x_p = 4k\cos 2t$. So $4k = 3$ and $k = \frac{3}{4}$. Then $x = A\cos 2t + B\sin 2t + \frac{3}{4}t\sin 2t$; $x(0) = 0$ gives $A = 0$; $x'(0) = 2B = 0$ gives $B = 0$. So $x = \frac{3}{4}t\sin 2t$.

d. The factor t makes the amplitude **grow without limit** (linearly in time). This is **resonance**: the forcing frequency equals the natural frequency. Engineers change the stiffness, add dampers, or alter the deck so that the natural frequency is away from walking frequencies.

Answer to Question 18

a. $0.0004\,I(1000 - I) = 0.0004 \times 1000\,I\left(1 - \dfrac{I}{1000}\right) = 0.4I\left(1 - \dfrac{I}{1000}\right)$.

b. This is logistic with $r = 0.4$ and capacity 1 000, so $I = \dfrac{1000}{1 + Ae^{-0.4t}}$. From $I(0) = 5$: $1 + A = 200$, so $A = 199$.

c. $I = 500$ when $199e^{-0.4t} = 1$, so $t = \dfrac{\ln 199}{0.4} \approx$ **13.2 days**.

d. The rate is $0.0004 \times 500 \times 500 =$ **100 students per day**. The product $I(1000 - I)$ is largest when $I = 500$ (a quadratic with maximum at the midpoint), so this is the fastest rate.

Answer to Question 19

a. Divide both sides by 2: $\dfrac{dh}{dt} = -0.001\sqrt{19.6h}$, and $\sqrt{19.6h} = \sqrt{19.6}\sqrt{h}$.

b. Separate: $\int h^{-1/2}dh = -0.001\sqrt{19.6}\int dt$, so $2\sqrt{h} = -0.001\sqrt{19.6}\,t + c$. At $t = 0$, $2\sqrt{4} = 4 = c$. So $\sqrt{h} = 2 - 0.0005\sqrt{19.6}\,t \approx 2 - 0.002214t$.

c. The tank is empty when $\sqrt{h} = 0$: $t = \dfrac{2}{0.0005\sqrt{19.6}} \approx 903.5$ s $\approx$ **15 minutes**.

d. When $h = 1$, $\sqrt{h} = 1$, so $0.0005\sqrt{19.6}\,t = 1$ and $t \approx 451.8$ s, about **7.5 minutes**. Half the time empties the top three quarters of the height, because the flow slows as the water level falls.

Answer to Question 20

a. With $y = vx$, the product rule gives $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. The right-hand side becomes $\dfrac{y}{x} + \dfrac{x}{y} = v + \dfrac{1}{v}$. Equating and cancelling v, we obtain $x\dfrac{dv}{dx} = \dfrac{1}{v}$.

b. This equation is separable: $\int v\,dv = \int \dfrac{dx}{x}$, so $\dfrac{v^2}{2} = \ln x + c$. Since $v = y/x$, $y^2 = 2x^2(\ln x + c)$. The condition $y(1) = 2$ gives $4 = 2c$, so $c = 2$. Taking the positive root (as $y > 0$), $y = x\sqrt{2\ln x + 4}$.

c. Since $\ln e = 1$, $y(e) = e\sqrt{2 + 4} = e\sqrt{6}$.

d. By the product and chain rules, $y' = \sqrt{2\ln x + 4} + \dfrac{1}{\sqrt{2\ln x + 4}}$. Also $\dfrac{y}{x} + \dfrac{x}{y} = \sqrt{2\ln x + 4} + \dfrac{1}{\sqrt{2\ln x + 4}}$. Both sides agree, so the solution is correct.