Test answers - Roots of polynomials, series and Maclaurin

Year: Further Maths · Duration: 60 minutes · /20

Answer to Question 1 (4 marks)

Method: for $ax^2+bx+c=0$, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$; for a cubic $ax^3+bx^2+cx+d=0$, $\sum\alpha=-\frac{b}{a}$, $\sum\alpha\beta=\frac{c}{a}$, $\alpha\beta\gamma=-\frac{d}{a}$. Other symmetric expressions are rewritten in terms of these, for example $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$. Mark scheme as shown in brackets.

a. Here $a=2$, $b=-3$, $c=4$, $d=-5$. $\sum\alpha=\frac{3}{2}$, $\sum\alpha\beta=2$ [1], $\alpha\beta\gamma=\frac{5}{2}$ [1]

b. Use $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$. $\left(\frac{3}{2}\right)^2-2\times2=-\frac{7}{4}$ [1]. A negative sum of squares means **the roots are not all real** [1]. Since the coefficients are real, there is one real root and a pair of complex conjugate roots.

Answer to Question 2 (4 marks)

Method: to transform the roots, set w equal to the new root in terms of x, make x the subject, and substitute into the original equation. Clear the fractions and expand. Mark scheme as shown in brackets.

Let $w=3x-1$, so $x=\frac{w+1}{3}$ [1]. Substitute: $\frac{(w+1)^3}{27}-\frac{2(w+1)^2}{9}+(w+1)-4=0$ [1]. Multiply by 27: $(w+1)^3-6(w+1)^2+27(w+1)-108=0$ [1], which expands to $w^3-3w^2+18w-86=0$ [1].

Check: the new sum of roots is $3\sum\alpha-3=3\times2-3=3$, which agrees with $-\frac{-3}{1}=3$.

Answer to Question 3 (4 marks)

Method: split the sum into multiples of $\sum r^3$, $\sum r^2$, $\sum r$ and $\sum 1=n$, substitute the standard formulae, and factorise out common factors such as $n(n+1)$ as early as possible. Mark scheme as shown in brackets.

a. $\frac{1}{4}n^2(n+1)^2-\frac{1}{2}n(n+1)$ [1] $=\frac{1}{4}n(n+1)\left(n(n+1)-2\right)$ [1]
$=\frac{1}{4}n(n+1)(n^2+n-2)=\frac{1}{4}(n-1)n(n+1)(n+2)$ ✓ [1]

b. $(r-1)r(r+1)=r^3-r$, so the sum is $\frac{1}{4}\times9\times10\times11\times12=$ **2970** [1]

Check with $n=2$: $(1-1)+(8-2)=6$ and $\frac{1}{4}\times1\times2\times3\times4=6$.

In (b), the sum starts with the term 0 when $r=1$.

Answer to Question 4 (4 marks)

Method: the Maclaurin series is $f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$. Often it is quicker to substitute into a known series (for e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^n$) and to keep the range of validity of that series. Mark scheme as shown in brackets.

a. Replace x by $2x$ in $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ [1]: $2x - 2x^2 + \frac{8}{3}x^3 - \cdots$ [1], valid for $-1 < 2x \leq 1$, i.e. $-\frac{1}{2} < x \leq \frac{1}{2}$ [1]

b. $0.2 - 0.02 + 0.002667 =$ **0.1827** [1]

The calculator gives $\ln 1.2 = 0.18232$, so three terms give an error of about 0.0004; the next term, $-4x^4 = -0.0004$, would correct it.

Answer to Question 5 (4 marks)

Method (differences): write the general term as $f(r) - f(r+1)$ using partial fractions. When the terms are added, everything cancels except the first part of the first term and the last part of the last term. Mark scheme as shown in brackets.

a. $\dfrac{60}{r(r+1)} = 60\left(\dfrac{1}{r} - \dfrac{1}{r+1}\right)$ [1]; the terms telescope to $60\left(1 - \dfrac{1}{n+1}\right) = \dfrac{60n}{n+1}$ ✓ [1]

b. $\dfrac{60n}{n+1} < 60$ for every n because $n < n+1$ [1]. $\dfrac{60n}{n+1} > 55$ gives $5n > 55$, $n > 11$: **after week 12** [1].

So the appeal raises more than £55 000 from week 12 onwards, but the total only approaches £60 000.