Lesson - Roots of polynomials, series and Maclaurin

Year: Further Maths

This chapter links the roots of a polynomial to its coefficients, sums standard series by formula and by the method of differences, and builds Maclaurin series for functions.

Starter activity

Think of a quadratic equation such as $x^2 - 7x + 10 = 0$. Solve it, then add the two roots and multiply them. Do the same for $x^2 + 3x - 18 = 0$. What do you notice about the sum and the product compared with the coefficients? Now try to guess what happens for a cubic like $(x-1)(x-2)(x-4) = 0$ once it is expanded. In the second half of this chapter we ask a related question in reverse: if we know the values of a function and its derivatives at a single point, how much of the function can we rebuild as a polynomial?

Roots and coefficients

Rule: Vieta's formulae

For $ax^2 + bx + c = 0$ with roots α, β: $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

For $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\dfrac{b}{a}$, $\sum \alpha\beta = \dfrac{c}{a}$, $\alpha\beta\gamma = -\dfrac{d}{a}$.

For $ax^4 + bx^3 + cx^2 + dx + e = 0$: $\sum \alpha = -\dfrac{b}{a}$, $\sum \alpha\beta = \dfrac{c}{a}$, $\sum \alpha\beta\gamma = -\dfrac{d}{a}$, $\alpha\beta\gamma\delta = \dfrac{e}{a}$.

The signs alternate.

Key fact: symmetric functions

$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$ and $\sum \dfrac{1}{\alpha} = \dfrac{\sum \alpha\beta}{\alpha\beta\gamma}$.

Example 1

The roots of $x^3 - 4x^2 + x + 6 = 0$ are α, β, γ. Find $\sum \alpha^2$.

$\sum \alpha = 4$, $\sum \alpha\beta = 1$, $\alpha\beta\gamma = -6$. So $\sum \alpha^2 = 4^2 - 2(1) = 14$.

Method: equation with transformed roots

1. Write $y = g(x)$ for the new roots.

2. Solve for x in terms of y.
3. Substitute into the original equation and simplify.
4. Rename y as x.

For roots $\alpha + 1$ put $x = y - 1$; for 2α put $x = \frac{y}{2}$; for $\frac{1}{\alpha}$ put $x = \frac{1}{y}$ and multiply by y^n.

Example 2

The roots of $x^3 - 6x^2 + 11x - 6 = 0$ are $1, 2, 3$. Find the equation with roots $\alpha - 1$.

Put $x = y + 1$: $(y+1)^3 - 6(y+1)^2 + 11(y+1) - 6 = y^3 - 3y^2 + 2y = 0$, with roots $0, 1, 2$ as expected.

Note: special cases

Roots in arithmetic progression $a - d, a, a + d$ give a from the sum of roots. Roots in geometric progression $\frac{a}{r}, a, ar$ give a^3 from the product. If the coefficients are real, a complex root brings its conjugate.

Watch out

Sign errors in Vieta's formulae are common: for a cubic, $\alpha\beta\gamma = -d/a$. Also remember to rename y back to x in a transformed equation.

Sums of series

Rule: standard sums

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

To sum from $r = a$ to b, subtract: $\sum_1^b - \sum_1^{a-1}$. Always factorise the answer.

Example 3

Find $\sum_{r=1}^{n} r(r+3)$.

$\frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) = \frac{1}{6}n(n+1)(2n+10) = \frac{1}{3}n(n+1)(n+5)$.

Example 4

Find $\sum_{r=1}^{n}(3r^2 - 1)$ and test it.

$\frac{1}{2}n(n+1)(2n+1) - n = \frac{n}{2}[(n+1)(2n+1) - 2] = \frac{1}{2}n(2n^2 + 3n - 1)$. For $n = 2$ the

sum is $2 + 11 = 13$ and the formula gives $\frac{1}{2} \cdot 2 \cdot 13 = 13$.

Method of differences

Definition: method of differences

Write the general term as $f(r) - f(r+1)$ (or $f(r+1) - f(r)$). The series then **telescopes**: all but a few terms cancel.

Method: telescoping a sum

1. Use partial fractions to write the general term as a difference.
2. Write out the first three and last three terms.
3. Cancel and list the terms that survive at each end.
4. For the sum to infinity, let $n \to \infty$.

Example 5

Find $\sum_{r=1}^{n} \frac{1}{r(r+1)}$ and the sum to infinity.

$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$, so $\sum_{r=1}^{n} = 1 - \frac{1}{n+1}$, and the sum to infinity is 1. For $\frac{1}{(r+1)(r+3)} = \frac{1}{2}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$ two terms survive at each end.

Maclaurin series

Theorem: Maclaurin series

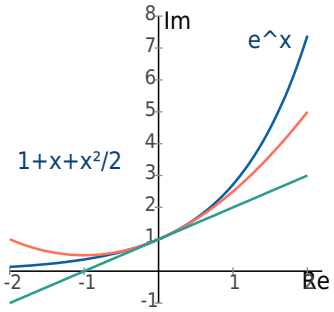
If f and its derivatives exist at 0:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Key fact: standard series

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (all x)
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ (all x)
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ (-1)

- $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \cdots$ ($|x|$ unless n is a positive integer)



Blue: y = e^x. Coral: quadratic Maclaurin polynomial. Green: linear approximation. Higher degree gives a better fit near x = 0.

Method: series of a composite or a solution of an equation

1. Substitute $x \to kx$ or $x \to x^2$ into a standard series; scale the range of validity (for example $\ln(1+2x)$ is valid for $-\frac{1}{2}$).
2. For a product, multiply the series and collect equal powers.
3. If y satisfies a differential equation, differentiate the equation repeatedly and evaluate at $x = 0$.

Example 6

Find the first two non-zero terms of the Maclaurin series of $\tan x$.

$y' = 1 + y^2$, $y'' = 2yy'$, $y''' = 2y'^2 + 2yy''$. At 0: $y = 0$, $y' = 1$, $y'' = 0$, $y''' = 2$. So $\tan x = x + \dfrac{2}{3!}x^3 + \cdots = x + \dfrac{x^3}{3} + \cdots$.

Example 7

Use a series to find $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ and approximate $\sqrt{1.1}$.

$1 - \cos x = \frac{x^2}{2} - \cdots$, so the limit is $\frac{1}{2}$. And $\sqrt{1.1} = (1+0.1)^{1/2} \approx 1 + 0.05 - 0.00125 + 0.0000625 = 1.0488125$.

Watch out

- Missing the surviving terms in a method of differences: list enough terms.
- Omitting the factorials in a Maclaurin series.
- Quoting a series without its range of validity.

More methods and worked examples

Method: finding symmetric functions of the roots

1. Write down $\sum\alpha$, $\sum\alpha\beta$, $\sum\alpha\beta\gamma$ and the product of the roots from the coefficients, taking care with the alternating signs.
2. Express the required quantity in terms of these sums. Useful identities are $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$ and $\sum\frac{1}{\alpha} = \frac{\sum\alpha\beta}{\alpha\beta\gamma}$ for a cubic.
3. For higher powers, use the fact that each root satisfies the equation, for example $\alpha^3 = -b\alpha^2 - c\alpha - d$ for a monic cubic, and add over all the roots.
4. Substitute the numerical values and simplify. Check, if possible, with roots that you can find directly.

Worked example 8: sum of reciprocals of squares

The roots of $x^3 - 4x^2 + 2x + 3 = 0$ are α, β, γ. Find $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

From the coefficients, $\sum\alpha = 4$, $\sum\alpha\beta = 2$ and $\alpha\beta\gamma = -3$. The reciprocals $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ have sum $\frac{\sum\alpha\beta}{\alpha\beta\gamma} = -\frac{2}{3}$ and pairwise products with sum $\frac{\sum\alpha}{\alpha\beta\gamma} = -\frac{4}{3}$. So the required sum of squares is $\left(-\frac{2}{3}\right)^2 - 2\left(-\frac{4}{3}\right) = \frac{4}{9} + \frac{8}{3} = \frac{28}{9}$. Treating the reciprocals as new roots in this way avoids messy fractions.

Worked example 9: a sum using the standard results

Find $\sum_{r=1}^{n}(2r-1)^3$ in a factorised form.

Expand: $(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$. Using the standard sums, the total is $8\cdot\frac{1}{4}n^2(n+1)^2 - 12\cdot\frac{1}{6}n(n+1)(2n+1) + 6\cdot\frac{1}{2}n(n+1) - n$. This simplifies to $2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n = n^2(2n^2-1)$. Check with $n=2$: $1+27 = 28$ and $4\times 7 = 28$. Always test a formula of this kind with a small value of n, because a slip in the algebra is easy to make and very easy to detect.

Worked example 10: a Maclaurin series by substitution

Find the first three non-zero terms of the series for $e^x\cos x$.

Multiply the series $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ and $1 - \frac{x^2}{2} + \cdots$, keeping only terms up to x^3. The constant is 1, the x term is x, the x^2 terms are $\frac{x^2}{2} - \frac{x^2}{2} = 0$, and the x^3 terms are $\frac{x^3}{6} - \frac{x^3}{2} = -\frac{x^3}{3}$. So $e^x\cos x = 1 + x - \frac{1}{3}x^3 + \cdots$. Multiplying known series is much quicker than differentiating the product three times, and it gives the same result.

Watch out: signs and validity

The signs in the formulae for the roots alternate: for $ax^3 + bx^2 + cx + d = 0$ the sum of the roots is $-\frac{b}{a}$, the sum of pairs is $+\frac{c}{a}$ and the product is $-\frac{d}{a}$. For a quartic the product of the four roots is $+\frac{e}{a}$. When you use a Maclaurin series, always state where it is valid: the series for e^x, $\sin x$ and $\cos x$ hold for all x, but the series for $\ln(1+x)$ and $(1+x)^n$ need -1 or $|x|$. A series used outside its interval of validity gives nonsense, however many terms you take.

How good is a Maclaurin approximation?

A Maclaurin polynomial matches the value of the function and of its first few derivatives at $x = 0$, so it is extremely accurate close to zero and less accurate further away. For an alternating series whose terms decrease in size, the error is smaller than the first term left out, which gives a simple way to decide how many terms you need. This is why small-angle approximations such as $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$ are used in physics and engineering: for angles of a few degrees the next terms are smaller than the errors of measurement.

Worked example 11: a limit using series

Find $\lim_{x \to 0} \dfrac{\ln(1+x) - x}{x^2}$.

Replace $\ln(1+x)$ by its series: $\ln(1+x) - x = -\frac{x^2}{2} + \frac{x^3}{3} - \cdots$. Dividing by x^2 gives $-\frac{1}{2} + \frac{x}{3} - \cdots$, and as x tends to zero every term except the first disappears. So the limit is $-\frac{1}{2}$. This method is often quicker and safer than other approaches, because the series shows exactly which power of x dominates near zero. A quick numerical check with $x = 0.01$ gives $\dfrac{0.00995033 - 0.01}{0.0001} = -0.4967$, which is close to -0.5 as expected.

Exam technique for roots and series

In questions on roots, examiners usually say "without solving the equation", so you must use the relations between roots and coefficients even if you can see the roots. Show the identity you use, such as the expression for the sum of squares, before substituting numbers. In questions on the method of differences, write out enough terms at the start and at the end of the sum to show clearly which terms cancel, because this display earns a method mark. In questions on Maclaurin series, show the derivatives evaluated at zero when the question asks you to find the series from first principles, and state the general term or the interval of validity when it is requested. Finally, simplify coefficients fully: a factorial left in the denominator, such as $\frac{8}{3!}$, should be written as $\frac{4}{3}$.

Connecting the three topics

The three parts of this chapter are closer than they first appear. The relations between roots and coefficients tell us about a polynomial from its coefficients, while a Maclaurin series builds a polynomial whose coefficients come from the derivatives of a function at a single point. Sums of series appear in both directions: the standard sums of powers are themselves polynomials in n, and the method of differences is a discrete version of the fundamental theorem of calculus, where a sum of differences collapses to the difference of the end values in the same way as an integral of a derivative does. Seeing these links will help you to choose the right method in a long problem.

Did you know?

The relations between the roots and the coefficients of a polynomial are named after the French lawyer and mathematician François Viète, who found them in about 1600. The series that carry the name of the Scottish mathematician Colin Maclaurin were published in 1742, as a special case of the series of Brook Taylor. Long before that, mathematicians in Kerala in southern India, led by Madhava around 1400, had already found the series for sine, cosine and arctangent, and used them to calculate π to eleven decimal places.

Otto's tip

Test a general formula with $n = 1$ and $n = 2$. If it fails on a small case you have an algebra slip.